Drinfel'd-Ihara Relations for the Crystalline Frobenius

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Notation and convention. If we have an element a of the ring of associative formal power series over a ring A, in the variables x_i, $i \in I$, we denote the coefficient of x^J in a, for a multi-index J, by $a[x^J]$. By a variety X over a field K, we mean a geometrically integral K-scheme X, that is separated and of finite type over K.

1. Introduction

For K a field of characteristic zero, let $\mathrm{M}(K)$ denote the set of Drinfel'd associators defined over K [Dr]. The variety M is of great interest because of its connection to the deformations of universal enveloping algebras, fundamental group of the Teichmuller tower, and to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. There is a natural map $\mathrm{M} \to \mathbb{A}^1$ and for $\lambda \in K$, and M_λ is a torsor under $\mathrm{GRT}_1\,(= \mathrm{M}_0)$. If $K << X, Y >>$ denotes the formal associative power series in X and Y over K then $\mathrm{GRT}_1(K)$ is defined to be the set of elements $\varphi \in K << X, Y >>$ that satisfy

$$
\begin{aligned}
&(i) \qquad \varphi(X,Y)\cdot\varphi(X,Y) = 1\\
&(ii) \qquad \varphi(Z,X)\cdot\varphi(Y,Z)\cdot\varphi(X,Y) = 1, \ \text{ when } \ X+Y+Z=0\\
&(iii) \quad \varphi(x_{23},x_{34})\cdot\varphi(x_{40},x_{01})\cdot\varphi(x_{12},x_{23})\cdot\varphi(x_{34},x_{40})\cdot\varphi(x_{01},x_{12}) = 1,\\
&\qquad\quad \text{when } x_{ii}=0,\ \sum_j x_{ij}=0,\ \text{and } [x_{ij},x_{kl}]=0,\ \text{if } \{i,j\}\cap\{k,l\}=\emptyset.
\end{aligned}
$$

The multi-zeta values (or equivalently the K-Z equation) gives an element in $\mathrm{GRT}_1(\mathbb{C})$ [Dr], the Galois action on $\pi_{1,et}(\mathbb{P}^1\backslash\{0,1,\infty\},\cdot)$ gives an element in $\mathrm{GRT}_1(\mathbb{Q}_\ell)$ [Ih]. The aim of the following is to show that the crystalline frobenius on $\pi_{1,dR}(\mathbb{P}^1\backslash\{0,1,\infty\},\cdot)$ also gives an element $\mathrm{GRT}_1(\mathbb{Q}_p)$. The main part is the proof of (iii) which is an application of the theory of tangential basepoints. Therefore most of the following is devoted to developing the crystalline theory of tangential basepoints. The Betti, de Rham, and étale realizations of the tangential basepoints in the case of curves were given in the fundamental paper of Deligne [De].

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2. Pullback to the log point

Let $\overline{X}/k$ be a smooth variety over a field k. Let $D \subseteq \overline{X}$ be a simple normal crossings divisor, $X := \overline{X}\backslash D$ and $x \in D(k)$. Let $\{D_i\}_{i\in I}$, be the set of irreducible components of D passing through x, $D_x := \cup_{i\in I} D_i$, I_i the ideal of D_i in $\mathcal{O}_{X,x}$, and $d : \mathfrak{m}_x \to \mathfrak{m}_x/\mathfrak{m}_x^2$ the canonical projection. Note that $I_i \subseteq \mathfrak{m}_x$. Denote by $\overline{X}_{log}$ the canonical fine saturated log scheme (in the Zariski topology) defined by $(\overline{X}, D)$. The underlying scheme of $\overline{X}_{log}$ is $\overline{X}$ and the log structure is defined by the inclusion

$$M_{\overline{X}} := \mathcal{O}_{\overline{X}} \cap i_*(\mathcal{O}_X^*) \to \mathcal{O}_{\overline{X}},$$

where $i : X \hookrightarrow \overline{X}$ is the inclusion. Note that

$$\overline{M}_{\overline{X},x} := M_{\overline{X},x}/\mathcal{O}^*_{\overline{X},x} \simeq \mathrm{Cart}^-(\overline{X}, D_x),$$

canonically, where $\mathrm{Cart}^-(\overline{X}, D_x)$ denotes the monoid of anti-effective Cartier divisors on $\overline{X}$ supported on D_x. If the D_i are defined locally by $t_i = 0$, with $t_i \in \mathcal{O}_{\overline{X},x}$, then $M_{\overline{X},x} = (\mathcal{O}_{\overline{X},x})^*_{\prod t_i}$.

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Let x_{log} denote the log scheme obtained by pulling back the log structure on $\overline{X}_{log}$ via the map $\mathrm{Spec}k \to \overline{X}$ corresponding to x. Note that the monoid $M_{x_{log}}$ on x_{log} is $M_{\overline{X},x} \otimes_{\mathcal{O}^*_{\overline{X},x}} k^*$.

Notation. With the notation above let

$$C(D_x, \overline{X}) := \{\varphi : \varphi \text{ is a splitting of } M_{x_{log}} \to \mathrm{Cart}^-(\overline{X}, D_x)\}.$$

Then

$$C(D_x, \overline{X}) \simeq \{(\cdots, \bar{t}_i, \cdots) : \bar{t}_i \in dI_i \setminus \{0\}, \ \text{ for } \ i \in I\}.$$

We say that $(\cdots, \bar{t}_i, \cdots)$ is *transversal* to D_x if $(\cdots, \bar{t}_i, \cdots) \in C(D_x, \overline{X})$. Similarly let

$$N_{D_x/\overline{X}} := \prod_{i \in I} N_{D_i/\overline{X}}(x),$$

be the fiber at x of the product of the normal bundles of D_i in $\overline{X}$. Note that elements of $C(D_x, \overline{X})$ are in one to one correspondence with linear isomorphisms

$$\prod_{i \in I} N_{D_i/\overline{X}}(x) \simeq \prod_{i \in I} \mathbb{A}^1$$

preserving the factors, and hence with $N^*_{D_x/\overline{X}}(k)$, where we let $N^*_{D_x/\overline{X}} := \prod_{i \in I} N^*_{D_i/\overline{X}}$, with $N^*_{D_i/\overline{X}} := N_{D_i/\overline{X}} \setminus \{0\}$. We call elements $(\cdots, v_i, \cdots) \in N^*_{D_x/\overline{X}}(k)$, r-tuples of vectors transversal to D_x. Therefore

Lemma 1. *The set of splittings of the log structure on x_{log} is in one to one correspondence with $N^*_{D_x/\overline{X}}(k)$.*

Let $\mathrm{Spec}\, k_{x,log}$ denote the log scheme with underlying scheme $\mathrm{Spec}\, k$, and log structure associated to the pre-log structure $\mathrm{Cart}^-(\overline{X}, D_x) \to k$ that maps all the nonzero elements of $\mathrm{Cart}^-(\overline{X}, D_x)$ to 0. From the lemma above the set of isomorphisms $x_{log} \simeq \mathrm{Spec}k_{x,log}$ is in one to one correspondence with $N^*_{D_x/\overline{X}}(k)$.

3. Construction of tangential basepoints in the logarithmic case

3.1. Crystalline. Let k be a perfect field of characteristic p. Let $W := W(k)$ be the ring of Witt vectors over k, with field of fractions K, and let $S := \mathrm{Spec}W$. Associated to the exact closed immersion

$$x_{log} \to \overline{X}_{log}$$

we have a pullback functor

$$x^*_{log} : \mathrm{Isoc}(\overline{X}_{log}/S) \to \mathrm{Isoc}(x_{log}/S)$$

between the categories of convergent log isocrystals on $\overline{X}_{log}$ and x_{log}. Choosing an element v in $N^*_{D/\overline{X}}(k)$ gives a functor

$$v^* : \mathrm{Isoc}(\overline{X}_{log}/S) \to \mathrm{Isoc}(\mathrm{Spec}\, k_{x,log}/S).$$

Definition 1. *Let $\mathcal{T}_x/K$ denote the tannakian category of vector bundles V over K endowed with a homomorphism $\varphi : \mathrm{Cart}^-(\overline{X}, D_x)^{gp\,*} \to \mathrm{End}(V)$, where*

$$\mathrm{Cart}^-(\overline{X}, D_x)^{gp\,*} := \mathrm{Hom}_{\mathbb{N}}(\mathrm{Cart}^-(\overline{X}, D_x), \mathbb{Z}).$$

The tensor product of (V, φ) and (W, ψ) is defined to be $V \otimes W$ endowed with the map that sends $-D_i \in \mathrm{Weil}^-(\overline{X}, D_x)$ to $id \otimes \psi(-D_i) + \varphi(-D_i) \otimes id \in \mathrm{End}(V \otimes W)$.

There is a natural realization functor

$$ev : \mathrm{Isoc}(\mathrm{Spec}\, k_{x,log}/S) \to \mathcal{T}_x$$

defined as follows. Let $S_{x,log}$ denote $\mathrm{Spec}W[\mathrm{Cart}^-(\overline{X}, D_x)]$ with the standard log structure. Note that $S_{x,log}/S$ is smooth and there is a canonical exact closed immersion

$$\mathrm{Spec}k_{x,log} \to S_{x,log}.$$

If $(E,\nabla) \in \mathrm{Isoc}(\mathrm{Spec}k_{x,log}/S)$ then we have the realization

$$(E,\nabla)_{S_{x,log}} \in \mathrm{Mic}(S_{x,log,K}/K).$$

The connection induces a map

$$-R : E(0) \to E(0) \otimes \Omega^1_{S_{x,log,K}/K}(0).$$

Since $\Omega^1_{S_{x,log,K}/K}(0) \simeq \mathrm{Cart}^-(\overline{X}, D_x)^{gp} \otimes_{\mathbb{Z}} K$, we obtain an object

$$((E,\nabla)_{S_{x,log,K}}(0), R) \in \mathcal{T}_x/K$$

This defines the functor above. We let $ev(v) := ev \circ v^*$, and $ev(v)$ composed with the natural fiber functor of $\mathcal{T}_x$ to be

$$fib(v) : \mathrm{Isoc}(\overline{X}_{log}/W) \to Vec_K$$

over K, for each $v \in N^*_{D/\overline{X}}(k)$, where Vec_K denotes the category of finite dimensional vector spaces over K.

3.2. de Rham. Let k be a field of characteristic zero. Then we have a canonical functor, which does not depend on the choice of a set of vectors transversal to D,

$$ev_{dR} : \mathrm{Mic}(x_{log}/k) \to \mathcal{T}_x/k$$

defined as follows. Let $(E,\nabla) \in \mathrm{Mic}(x_{log}/k)$. Then as above we have a residue map

$$R : E \to E \otimes \Omega^1_{x_{log}/k},$$

and an isomorphism $\Omega^1_{x_{log}/k} \simeq \mathrm{Cart}^-(\overline{X}, D_x)^{gp} \otimes_{\mathbb{Z}} k$ which gives an object of $\mathcal{T}_x$ as above. For $v \in N^*_{D_x/\overline{X}}(k)$ we obtain a commutative diagram

$$\begin{array}{ccc}
\mathrm{Mic}(x_{log}/k) & \xrightarrow{ev_{dR}} & \mathcal{T}_x \\
\big\downarrow{\scriptstyle id_v} & & \big\downarrow{\scriptstyle id} \\
\mathrm{Mic}(k_{x,log}/k) & \xrightarrow{ev_{dR}} & \mathcal{T}_x,
\end{array}$$

where id_v denotes the identification as above that depends on the choice of a transversal set of tangent vectors; and where the lower horizontal map is defined analogously to the horizontal map above. Let $ev_{dR}(x) := ev_{dR} \circ x^*_{log}$ and $ev_{dR}(v) := ev_{dR} \circ id_v \circ x^*_{log}$; and $fib_{dR}(\cdot)$, $e_{dR}(\cdot)$ composed with the standard fiber functor of Vec_k. Then by the above we have a canonical isomorphism $fib_{dR}(x) \simeq fib_{dR}(v)$ and in particular, in the logarithmic de Rham theory the fiber functor is independent of the choice of vectors.

3.3. Comparison. With the notation as in the beginning of this section, assume that k is a perfect field of characteristic p. Fix $v \in N^*_{D_x/\overline{X}}(k)$, a vector transversal to D_x. Let $\overline{\mathfrak{X}}/W$ be a smooth formal scheme, $\mathfrak{D} \subseteq \overline{\mathfrak{X}}$ a simple relative normal crossings divisor, $\mathfrak{x} \in \mathfrak{D}(W)$ such that the reductions of $\overline{\mathfrak{X}}$, $\mathfrak{D}$, and $\mathfrak{x}$ are $\overline{X}$, D, and x respectively. We let $\mathfrak{X} := \overline{\mathfrak{X}} \setminus \mathfrak{D}$, $\overline{\mathfrak{X}}_{log}$ the log scheme associated to the pair $(\overline{\mathfrak{X}}, \mathfrak{D})$ etc. Then $N_{\mathfrak{D}_{\mathfrak{x}}/\overline{\mathfrak{X}}}$ is a lifting of $N_{D_x/\overline{X}}$. Let $\mathfrak{v} \in N^*_{\mathfrak{D}_{\mathfrak{x}}/\overline{\mathfrak{X}}}(W)$, with reduction v. We have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Spec}\, k_{x,log} & \longrightarrow & \mathrm{Spec}\, W_{\mathfrak{x},log} \\
\big\downarrow{\scriptstyle id_v} & & \big\downarrow{\scriptstyle id_{\mathfrak{v}}} \\
\mathrm{Spec}\, x_{log} & \longrightarrow & \mathrm{Spec}\, \mathfrak{x}_{log} \\
\big\downarrow & & \big\downarrow \\
\overline{X}_{log} & \longrightarrow & \overline{\mathfrak{X}}_{log},
\end{array}$$

and a canonical exact closed imbedding $\mathrm{Spec}W_{\mathfrak{x},log} \hookrightarrow \mathfrak{S}_{\mathfrak{x},log}$. Note that $\mathrm{Spec}W_{\mathfrak{x},log}$ does not depend on $\mathfrak{x}$, therefore we will denote it by $\mathrm{Spec}W_{x,log}$.

For $(E,\nabla) \in \mathrm{Isoc}(\overline{X}_{log}/W)$, let $(E,\nabla)_{\overline{\mathfrak{X}}} \in \mathrm{Mic}(\overline{\mathfrak{X}}_{K,log}/K)$ denote the realization of (E,∇) on the log rigid analytic space $\overline{\mathfrak{X}}_{K,log}$. Then since $id_{\mathfrak{v}}$ is the identity map on the underlying schemes,

we have $fib(v)(E,\nabla) \simeq E_{\overline{\mathfrak{X}}}(\mathfrak{x}_K)$ by the definition of $fib(v)$.We call this the *realization* of the fiber functor $fib(v)$ corresponding to the data $(\overline{\mathfrak{X}}, \mathfrak{D}, \mathfrak{x}, \mathfrak{v})$, and denote it by $fib(v)_{\mathfrak{v}}$ in order to remember the choice of the model. Note that this in fact does not depend on the choice of the lifting $\mathfrak{v}$ as expected from the de Rham version. However the comparison with the different models, i.e. the crystalline version, depends on the choice of the liftings of the tangent vectors. Fixing the model gives the comparison between the de Rham and the crystalline versions.

Let $(\overline{\mathfrak{Y}}, \mathfrak{E}, \mathfrak{y}, \mathfrak{u})$ be another data of a lifting. Let $(\overline{\mathfrak{X}} \times \overline{\mathfrak{Y}})^{\sim}$ denote the blow-up of $\overline{\mathfrak{X}} \times \overline{\mathfrak{Y}}$ along $\cup_i(\mathfrak{D}_i \times \mathfrak{E}_i)$. Denote $\mathfrak{D}_i \times \mathfrak{E}_i$ by $\mathfrak{F}_i$. Endow $(\overline{\mathfrak{X}} \times \overline{\mathfrak{Y}})^{\sim}$ with the log structure associated to the exceptional divisor of the blow-up, and denote it by $(\overline{\mathfrak{X}} \times \overline{\mathfrak{Y}})^{\sim}_{log}$. The fiber of the blow-up over the point $(\mathfrak{x}, \mathfrak{y})$ is isomorphic to

$$\times_i \mathbb{P}(N_{\mathfrak{F}_i/\overline{\mathfrak{X}} \times \overline{\mathfrak{Y}}}),$$

the product of the projective normal bundles of $\mathfrak{F}_i$ in $\overline{\mathfrak{X}} \times \overline{\mathfrak{Y}}$. In particular the pair $[\mathfrak{v}, \mathfrak{u}]$ defines an element of the fiber of

$$(\overline{\mathfrak{X}} \times \overline{\mathfrak{Y}})^{\sim} \to \overline{\mathfrak{X}} \times \overline{\mathfrak{Y}}$$

over $(\mathfrak{x}, \mathfrak{y})$, where if $\mathfrak{v} = (\cdots, \mathfrak{v}_i, \cdots)$, and $\mathfrak{u} = (\cdots, \mathfrak{u}_i, \cdots)$ then

$$[\mathfrak{v}, \mathfrak{u}] := (\cdots, [\mathfrak{v}_i, \mathfrak{u}_i], \cdots).$$

If $(E,\nabla) \in \mathrm{Isoc}(\overline{X}_{log}/W)$ then we have a canonical isomorphism between the pull-backs of $(E,\nabla)_{\overline{\mathfrak{X}}}$ and $(E,\nabla)_{\overline{\mathfrak{Y}}}$ to $(\overline{\mathfrak{X}} \times \overline{\mathfrak{Y}})^{\sim}_K$, and hence evaluating this isomorphism at $[\mathfrak{v}, \mathfrak{u}]_K$ gives an isomorphism

$$E_{\overline{\mathfrak{X}}}(\mathfrak{x}_K) \simeq E_{\overline{\mathfrak{Y}}}(\mathfrak{y}_K).$$

On the other hand from the above diagram applied to $\overline{\mathfrak{X}}, \mathfrak{x}, \mathfrak{v}$ and $\overline{\mathfrak{Y}}, \mathfrak{y}, \mathfrak{u}$, we obtain isomorphisms, functorial in E,

$$E_{\overline{\mathfrak{X}}}(\mathfrak{x}_K) \simeq fib(v)_{\mathfrak{v}}(E,\nabla) \simeq fib(v)_{\mathfrak{u}}(E,\nabla) \simeq E_{\overline{\mathfrak{Y}}}(\mathfrak{y}_K).$$

Claim. The two isomorphisms above, between $E_{\overline{\mathfrak{X}}}(\mathfrak{x}_K)$ and $E_{\overline{\mathfrak{Y}}}(\mathfrak{y}_K)$, are the same.

Proof. Let $M := M_{W_{x,log}}$, and $\Delta : M \oplus M \to M$, the map induced by (id_M, id_M), and I the ideal $\Delta^*((-\sum_{i \in I} D_i))$. Let $(W_{x,log} \times W_{x,log})^{\sim}$ denote the blow-up of $W_{x,log} \times W_{x,log}$ along I. Note that its underlying scheme is $\prod_{i \in I} \mathbb{P}^1_W$.

Let

$$\alpha_{\mathfrak{v}} : \mathrm{Spec} W_{x,log} \to \overline{\mathfrak{X}}$$

and

$$\alpha_{\mathfrak{u}} : \mathrm{Spec} W_{x,log} \to \overline{\mathfrak{Y}}$$

be the exact closed imbeddings induced by $\mathfrak{v}$ and $\mathfrak{u}$. These induce a map

$$(W_{x,log} \times W_{x,log})^{\sim} \to (\overline{\mathfrak{X}} \times \overline{\mathfrak{Y}})^{\sim}_{log}$$

that maps $(1, \cdots, 1)$ to $[\mathfrak{v}, \mathfrak{u}]$. The diagonal map factors through the blow-up to give a map $\Delta : \mathrm{Spec} W_{x,log} \to (W_{x,log} \times W_{x,log})^{\sim}$, which has image $(1, \cdots, 1)$. This map makes the diagrams

$$\begin{array}{ccccc} \mathrm{Spec} W_{x,log} & \longrightarrow & \mathrm{Spec}(W_{x,log} \times W_{x,log})^{\sim} & \longrightarrow & (\overline{\mathfrak{X}} \times \overline{\mathfrak{Y}})^{\sim}_{log} \\ \big\downarrow{\scriptstyle id} & & & & \big\downarrow \\ \mathrm{Spec} W_{x,log} & & \xrightarrow{\alpha_{\mathfrak{v}}} & & \overline{\mathfrak{X}}_{log} \end{array}$$

and

$$\begin{array}{ccccc} \mathrm{Spec} W_{x,log} & \longrightarrow & \mathrm{Spec}(W_{x,log} \times W_{x,log})^{\sim} & \longrightarrow & (\overline{\mathfrak{X}} \times \overline{\mathfrak{Y}})^{\sim}_{log} \\ \big\downarrow{\scriptstyle id} & & & & \big\downarrow \\ \mathrm{Spec} W_{x,log} & & \xrightarrow{\alpha_{\mathfrak{u}}} & & \overline{\mathfrak{Y}}_{log} \end{array}$$

commute. Therefore if $(E,\nabla) \in \mathrm{Isoc}(\overline{X}_{log}/W)$ then the isomorphism between

$$fib(v)_{\mathfrak{v}}(E,\nabla) \quad \text{and} \quad fib(v)_{\mathfrak{u}}(E,\nabla)$$

is obtained by pulling back the canonical isomorphism between

$$p_1^*(E,\nabla)_{\overline{\mathfrak{X}}} \text{ and } p_2^*(E,\nabla)_{\overline{\mathfrak{Y}}}$$

via

$$\mathrm{Spec} W_{x,log} \to (\overline{\mathfrak{X}} \times \overline{\mathfrak{Y}})^{\sim}_{log},$$

where the p_i, $i = 1,2$ denote the projections. Since the last map has image $[\mathfrak{v},\mathfrak{u}]$, this proves the claim. $\square$

4. Comparison of different log points

Let $D_{(x)} \subseteq N_{D_x/\overline{X}}$ denote the simple normal crossings divisor defined by the ideal $\prod_{i\in I} dI_i$. In other words if $t_1,\cdots,t_n$ is a regular system of parameters on $\overline{X}$ at x, such that D is defined locally by $t_1\cdots t_r = 0$ then $D_{(x)}$ is defined by $dt_1\cdots dt_r = 0$. Note that $\{dt_i\}_{1\le i\le r}$ is a system of coordinates for $N_{D_x/\overline{X}}$. Let $N_{log} := N_{D_x/\overline{X},log}$ be the scheme $N_{D_x/\overline{X}}$ endowed with the logarithmic structure associated to the divisor $D_{(x)}$, and 0_{log} the logarithmic scheme induced by the inclusion $0 \to N_{D_x/\overline{X}}$ which gives an exact closed immersion

$$0_{log} \to N_{log}.$$

Lemma 2. *The log schemes x_{log} and 0_{log} are canonically isomorphic.*

Proof. Note that both log schemes have the same underlying scheme $\mathrm{Spec}\, k$.

Let $\{t_i\}_{i\in I}$ be a part of a system of parameters for $\mathfrak{m}_x$ such that D is defined locally by $t_1\cdots t_r = 0$. This gives a map

$$\varphi : \otimes_{1\le i\le r} Sym^{\cdot}(dI_i/dI_i^2) \to \mathcal{O}_{\overline{X},x},$$

with the property $\varphi(dt_i) = t_i$. Since $M_N = (\mathcal{O}_{N,0})^*_{dt_1\cdots dt_r}$, $M_{\overline{X},x} = (\mathcal{O}_{X,x})^*_{t_1\cdots t_r}$, this gives a map $M_N \to M_{\overline{X},x}$ and hence a map

$$M_T \otimes_{\mathcal{O}^*_{N,x}} k^* \to M_{\overline{X},x} \otimes_{\mathcal{O}^*_{\overline{X},x}} k^*$$

that we continue to denote by φ.

The last map is independent of the choice of $\{t_i\}_{1\le i\le r}$ as above. If $\{s_i\}_{i\in I}$ is another set then $s_i/t_i \in \mathcal{O}^*_{\overline{X},x}$, for $i \in I$. Let ψ be the map corresponding to $\{s_i\}_{i\in I}$. To show the independence it is enough to show that ψ and φ agree on $\{ds_i\}_{i\in I}$. Note that

$$\varphi(ds_i) = \varphi(d(\frac{s_i}{t_i}\cdot t_i)) = \varphi(d(\frac{s_i}{t_i})\cdot t_i + \frac{s_i}{t_i}(x)\cdot dt_i) = \frac{s_i}{t_i}(x)\cdot t_i$$

and $\psi(ds_i) = s_i$. Therefore

$$\frac{\varphi(ds_i)}{\psi(ds_i)} = \frac{s_i}{t_i}(x)\cdot\frac{t_i}{s_i}$$

and hence $\varphi(ds_i)$ and $\psi(ds_i)$ are equal in $M_{\overline{X},x} \otimes_{\mathcal{O}^*_{\overline{X},x}} k^*$, for $i \in I$. This shows the independence.

This map makes the diagram

$$\begin{tikzcd}
0 \arrow[r] & k^* \arrow[r] \arrow[d, "id"] & M_{0_{log}} \arrow[r] \arrow[d] & \overline{M}_{0_{log}} \arrow[r] \arrow[d, "\alpha_x"] & 0 \\
0 \arrow[r] & k^* \arrow[r] & M_{x_{log}} \arrow[r] & \overline{M}_{x_{log}} \arrow[r] & 0
\end{tikzcd}$$

commute, where α_x is the canonical isomorphism

$$\alpha_x : \mathrm{Cart}^-(N_{D_x/\overline{X}}, D_{(x)}) \simeq \mathrm{Weil}^-(N_{D_x/\overline{X}}, D_{(x)}) \to \mathrm{Weil}^-(\overline{X}, D_x) \simeq \mathrm{Cart}^-(\overline{X}, D_x).$$

This implies that the middle map is an isomorphism. $\square$

5. Tangential basepoints in the regular singular case

We continue with the notation above. Let $\hat{N}_{log}$ and $\hat{\overline{X}}_{log}$ denote the completions of N_{log} along 0 and $\overline{X}$ along x. Let $\psi : \hat{N}_{log} \to \hat{\overline{X}}_{log}$ be a morphism making the diagram

$$\begin{array}{ccc} 0_{log} & \longrightarrow & x_{log} \\ \downarrow & & \downarrow \\ \hat{N}_{log} & \xrightarrow{\psi} & \hat{\overline{X}}_{log} \end{array}$$

commute. Such a ψ can be obtained by choosing a system of parameters $t_1, \cdots, t_n$ at x on $\overline{X}$ such that D_x is locally defined by $t_1 \cdots t_r = 0$, and letting $\psi^*(t_i) = dt_i \cdot \varphi_i$ with $\varphi_i \in 1 + \hat{\mathfrak{m}}_0$, for $1 \leq i \leq r$, and $\psi^*(t_j) = s_j$, with $s_j \in \hat{\mathfrak{m}}_0$, for $r+1 \leq j \leq n$.

5.1. Construction. In this section assume that the base field K is of characteristic zero. We say that an integrable connection ∇ on a vector bundle E on $K[[t_1, \cdots, t_n]]_{t_1 \cdots t_r}$ is regular along the divisor $t_1 \cdots t_r = 0$, if (E, ∇) has a logarithmic extension $(\overline{E}, \nabla)$ to $K[[t_1, \cdots, t_n]]$. If Y/K is smooth, and $D \subseteq Y$ is a simple normal crossings divisor then we denote the category of vector bundles with integrable connection on $Y \setminus D$ with regular singularities along D by $\mathrm{Mic}_{reg,D}(Y/K)$.

We define a natural functor,

$$\hat{\varphi}^* : \mathrm{Mic}_{reg, \hat{D}_x}(\hat{\overline{X}}/K) \to \mathrm{Mic}_{reg, \hat{D}_{(x)}}(\hat{N}/K).$$

We would like to define $\hat{\varphi}^*$ as ψ^*. However since there are different choices for ψ, we have to give canonical isomorphisms between ψ^* and ψ'^* for two such choices ψ and ψ', satisfying the cocycle condition for three choices. This is done exactly as above. Let $(\hat{\overline{X}} \times \hat{\overline{X}})^\sim$ be the blow-up of $(\hat{\overline{X}} \times \hat{\overline{X}})$ along $\cup_{i \in I}(\hat{D}_i \times \hat{D}_i)$. Let $\hat{D}_x^\sim$ denote the exceptional divisor. Then ψ and ψ' induce a map $(\psi, \psi')^\sim : \hat{N}_{log} \to (\hat{\overline{X}} \times \hat{\overline{X}})^\sim_{log}$ that factors through the completion $\hat{\Delta}^\sim_{\overline{X}}$ of $(\hat{\overline{X}} \times \hat{\overline{X}})^\sim_{log}$ along the strict transform $\Delta^\sim_{\overline{X}}$ of the diagonal.

Let $(E, \nabla) \in \mathrm{Mic}_{reg, \hat{D}_x}(\hat{\overline{X}}/K)$ then it has a logarithmic extension $(\overline{E}, \nabla)$. The connection gives a horizontal isomorphism between $p_1^*(\overline{E}, \nabla)$ and $p_2^*(\overline{E}, \nabla)$ on the first infinitesimal neighborhood of the strict transform of the diagonal in $(\hat{\overline{X}} \times \hat{\overline{X}})^\sim$. Since the connection is integrable and K has characteristic 0 the isomorphism extends to the formal neighborhood. The restriction of this isomorphism to $(\hat{\overline{X}} \times \hat{\overline{X}})^\sim \setminus \hat{D}_x^\sim$ depends only on (E, ∇). Therefore pulling back this isomorphism via

$$\hat{N} \setminus \hat{D}_{(x)} \to (\hat{\overline{X}} \times \hat{\overline{X}})^\sim \setminus \hat{D}_x^\sim$$

gives the isomorphisms between ψ^* and ψ'^* we were looking for. As usual the integrability of the connection gives the cocycle condition.

We will need the following.

Lemma 3. *If $(E, \nabla) \in \mathrm{Mic}_{reg, t_1 \cdots t_r}(K[[t_1, \cdots, t_n]]/K)$ then it has an extension of the form $(\mathcal{O}^{\oplus rkE}, d - \sum_{1 \leq i \leq r} \Gamma_i \frac{dz_i}{z_i})$, to $K[[t_1, \cdots, t_n]]$, where $\Gamma_i \in \mathrm{M}_{rkE \times rkE}(K)$ and the Γ_i do not have eigenvalues that differ by a non-zero integer.*

Proof. This is exactly Théorème 3.4. in [GL] except that there K is assumed to be algebraically closed. The same proof works without this assumption as follows. Using the notation in loc. cit. let Λ be a lattice, not necessarily free, in $\Gamma(K[[t_1, \cdots, t_n]]_{t_1 \cdots t_r}, E)$ relative to which ∇ has logarithmic singularities along $t_1 \cdots t_r = 0$. Let W be a K-subspace of Λ invariant under the semi-simplifications ${}^s\nabla_{t_i \partial/\partial t_i}$ of the $\nabla_{t_i \partial/\partial t_i}$ and complementary to $\mathfrak{m}\Lambda$. Since ${}^s\nabla_{t_i \partial/\partial t_i}$ are commuting semi-simple operators, W has a decomposition $W = \oplus_{j \in J} W_j$, as a $K[{}^s\nabla_{t_i \partial/\partial t_i}]_{1 \leq i \leq n}$ module, such that each ${}^s\nabla_{t_i \partial/\partial t_i}|_{W_j}$ is annihilated by an irreducible polynomial. Assume that, ${}^s\nabla_{t_i \partial/\partial t_i}$, for some i, has two eigenvalues λ_1 and λ_2 such $\lambda_1 - \lambda_2 = n \in \mathbb{Z}^+$. Note that these eigenvalues cannot be the root of the same irreducible polynomial $p(x)$, since otherwise λ_1 will be a root of $p(x) - p(x-n)$, a non-zero polynomial of degree less than $p(x)$. If λ_k, $1 \leq k \leq 2$, appear as an eigenvalue of ${}^s\nabla_{t_i \partial/\partial t_i}$

on W_{j_k}, then the corresponding minimal polynomials satisfy $p_{i_1}(x) = p_{i_2}(x-n)$. We replace W_{i_2} with $t_i^n W_{i_2}$, in $W = \sum_{j\in J} W_j$, and continue in this manner until no two eigenvalues for the ${}^s\nabla_{t_i\partial/\partial t_i}$ differ by a non-zero integer, and denote by Λ' the lattice generated by this. Then Théorème 3.4. in loc. cit. shows that Λ' is free, invariant under the $\nabla_{t_i\partial/\partial t_i}$, and ∇ is represented on Λ' as in the statement of the lemma. □

Lemma 4. *$\hat{\varphi}^*$ is an equivalence of categories.*

Proof. In order to prove this, by the definition above, we may assume that $\hat{\varphi}^*$ is i^*, where i is the closed immersion

$$i : \mathrm{Spf} K[[t_1, \cdots, t_r]]_{t_1\cdots t_r} \to \mathrm{Spf} K[[t_1, \cdots, t_n]]_{t_1\cdots t_r}.$$

Since i^* has a section, namely pr^*, where pr is the projection, the essential surjectivity follows. To see the full-faithfullness it is enough to show, for

$$(E, \nabla) \in \mathrm{Mic}_{reg,t_1\cdots t_r}(K[[t_1, \cdots, t_n]]/K),$$

that

$$\Gamma_\nabla(K[[t_1, \cdots, t_n]]_{t_1\cdots t_r}, E) \to \Gamma_{i^*\nabla}(K[[t_1, \cdots, t_r]]_{t_1\cdots t_r}, i^*E)$$

is an isomorphism. By lemma 3. we may assume that

$$(E, \nabla) = (\mathcal{O}^{\oplus rkE}, d - \sum_{1\le i\le r} \Gamma_i \frac{dz_i}{z_i})$$

such that the only integer eigenvalues of the Γ_i are 0. Then we see that $U(z) := \sum_{I\in\mathbb{Z}^r} U_I z^I \in \Gamma_\nabla(K[[t_1, \cdots, t_n]]_{t_1\cdots t_r}, E)$ if and only if $U_0 \in \cap_{1\le i\le r}\Gamma_i$ and $U_I = 0$, for $I \in \mathbb{Z}^r \setminus \{0\}$. This shows the bijectivity. □

Finally in order to define the tangential basepoints we will need the following

Lemma 5. *Let $\mathrm{Mic}_{reg}(K[t_1, \cdots, t_n]_{t_1\cdots t_n}/K)$ denote the category of vector bundles with integrable connection and regular singularities at infinity on $(\mathbb{A}^1 \setminus \{0\})^n$. Then the natural map*

$$\mathrm{Mic}_{reg}(K[t_1, \cdots, t_n]_{t_1\cdots t_n}/K) \to \mathrm{Mic}_{reg,t_1\cdots t_r}(K[[t_1, \cdots, t_n]]/K)$$

is an equivalence of categories.

Proof. Since $(\mathcal{O}^m, d - \sum_{1\le i\le n} \Gamma_i \frac{dz_i}{z_i})$, with $\Gamma_i \in M_{m\times m}(K)$, has regular singularities along all the divisors at infinity, the essential surjectivity of the functor follows from lemma 3.

Let $(E, \nabla) \in \mathrm{Mic}_{reg}(K[t_1, \cdots, t_n]_{t_1\cdots t_n}/K)$ then since E has a locally free extension to $K[t_1, \cdots, t_n]$, e.g. a logarithmic extension of (E, ∇), the natural map

$$\Gamma_\nabla(K[t_1, \cdots, t_n]_{t_1\cdots t_n}, E) \to \Gamma_\nabla(K[[t_1, \cdots, t_n]]_{t_1\cdots t_n}, E)$$

is injective. To see its surjectivity, let s be in the target. Then (E, ∇) and s can be defined over a subfield K_0 of K that has countable transcendence degree over $\mathbb{Q}$. Therefore it suffices to prove the surjectivity where K is replaced by an arbitrary subfield L of $\mathbb{C}$. If $L = \mathbb{C}$ then by the Riemann-Hilbert correspondence we may assume that $(E, \nabla) = (\mathcal{O}^{\oplus k}, d - \sum_{1\le i\le n} \Gamma_i \frac{dz_i}{z_i})$, where $\Gamma_i \in M_{k\times k}(\mathbb{C})$ with 0 as the only possible integer eigenvalue for the Γ_i's. Then any formal solution of the differential equation around 0 is a constant section of $\mathcal{O}^k$ that is annihilated by all the Γ_i's. This gives the surjectivity when $L = \mathbb{C}$. For general $L \subseteq \mathbb{C}$, let $s \in \Gamma_\nabla(L[[t_1, \cdots, t_n]]_{t_1\cdots t_n}, E)$. Then, by the above, there is an $s' \in \Gamma_\nabla(\mathbb{C}[t_1, \cdots, t_n]_{t_1\cdots t_n}, E)$ such that s' maps to the pull-back of s by $\mathbb{C}/L$. Then by the injectivity proven above, s' is invariant under $\mathrm{Gal}(\mathbb{C}/L)$ since s is. Therefore by descent s comes from the variety over L. □

Therefore we obtain a quasi-inverse

$$\mathrm{Mic}_{reg,\hat{D}_{(x)}}(\hat{N}/K) \to \mathrm{Mic}_{reg}(N^*_{D_x/\overline{X}}/K),$$

and combining this with

$$\mathrm{Mic}_{reg}(X/K) \to \mathrm{Mic}_{reg,\hat{D}_{(x)}}(\hat{N}/K)$$

and $\hat{\varphi}^*$ gives

$$\mathrm{Mic}_{reg}(X/K) \to \mathrm{Mic}_{reg}(N^*_{D_x/\overline{X}}/K)$$

5.2. Functoriality. Let $f : \overline{X}_{log} \to \overline{Y}_{log}$ be a morphism of log schemes with $f(x) = y$. Let $\mathcal{P}(f) : N_{x,X,log} \to N_{y,Y,log}$ denote the unique homogeneous map that induces $f_x : x_{log} \to y_{log}$ under restriction and the isomorphisms $x_{log} \simeq 0_{x,log}$ and $y_{log} \simeq 0_{y,log}$. $\mathcal{P}(f)$ is called the *principal part* of f at x relative to the given divisors. Then we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Mic}_{reg,E}(Y/K) & \xrightarrow{f^*} & \mathrm{Mic}_{reg,D}(X/K) \\
\downarrow & & \downarrow \\
\mathrm{Mic}_{reg,E_{(y)}}(N_y/K) & \xrightarrow{\mathcal{P}(f)^*} & \mathrm{Mic}_{reg,D_{(x)}}(N_x/K).
\end{array}$$

6. The unipotent case

6.1. We will need the following in order to give the relation between tangential and ordinary basepoints. In a tannakian category we denote the tannakian subcategory of unipotent objects by the subscript *uni*.

Lemma 6. *Let $D_i \subseteq (\mathbb{P}^1_K)^r$, for $i \in \{0, \infty\}$ denote the divisor defined by the standard coordinate axes passing through i. And let $D := D_0 \cup D_\infty$. Then the restriction to the origin map*

$$\mathrm{Mic}_{uni}((\mathbb{P}^1_K)^r_{log}/K) \to \mathrm{Mic}_{uni}(K^r_{log}/K)$$

is an equivalence of categories.

Proof. Denote $(\mathbb{P}^1)^r$ by P. First note that $\mathrm{Mic}_{uni}(K^r_{log}/K)$ is canonically equivalent to $\mathcal{T}_{uni,r}/K$ the full subcategory of unipotent objects in $\mathcal{T}_r/K$, where by $\mathcal{T}_r$ we mean the category defined as in Definition. 1 above with $\mathrm{Cart}^-(\cdot)$ replaced with $\mathbb{N}^r$. Under this equivalence the functor above becomes the one that associates $E(0)$ endowed with $\{\mathrm{res}_i(E,\nabla)\}_{1\le i\le r}$ to (E,∇), where res_i denotes the residue along D_i at 0.

Let $(V, \{N_i\}_{i\in I})$ be an object of $\mathcal{T}_{r,uni}$ then $(V \otimes_K \mathcal{O}_P, d - \sum_{i\in I} N_i d\log z_i)$ is an object of $\mathrm{Mic}_{uni}(P_K/K)$ that has image $(V, \{N_i\}_{i\in I})$ under the restriction functor. This proves the essential surjectivity.

Since the functor above is a tensor functor in order to see that it is fully faithful it is enough to show that taking fibers at zero induces an isomorphism

$$\mathrm{Hom}_\nabla((\mathcal{O}_{P_K}, d), (E,\nabla)) \to \mathrm{Hom}_{\mathcal{T}_r}((K, \{0\}_{i\in I}), (E(0), \{\mathrm{res_i(E,\nabla)}\}_{i\in I})$$

is an isomorphism, or equivalently that

$$\mathrm{H}^0_{dR}(P_{K,log}, (E,\nabla)) \to \cap_{i\in I}\mathrm{ker}_{E(0)}(\mathrm{res}_i(E,\nabla))$$

is an isomorphism. First note that for the underlying bundle E of

$$(E,\nabla) \in \mathrm{Mic}_{uni}(P_K/K)$$

is trivial. This follows from the fact that $\mathrm{Ext}^1_P(\mathcal{O}_P, \mathcal{O}_P) = \mathrm{H}^1(P, \mathcal{O}_P) = 0$ by induction on the nilpotence level [De]. Therefore without loss of generality we will assume that $(E,\nabla) = (\mathcal{O}_P^{rkE}, d - \sum_{1\le i\le r} N_i\, d\log z_i)$, for some nilpotent matrices $N_i \in M_{rkE\times rkE}(K)$. If α is a global (horizontal) section of (E,∇) then it is a constant section of $\mathcal{O}^{rkE}$. Therefore the map above is injective. In order to see that it is surjective, we note that for any $\alpha \in \cap_{i\in I}\mathrm{ker}_{E(0)}(\mathrm{res}_i(E,\nabla))$, the constant section of $\mathcal{O}_P^{rkE}$ with fiber α at 0 is a horizontal section with respect to the connection $d - \sum_{i\in I} N_i d\log z_i$. □

Let $\overline{N}_{D_x/\overline{X}} := \prod_{i\in I} \overline{N}_{D_i/\overline{X}}$, where $\overline{N}_{D_i/\overline{X}}$ is the smooth compactification of $N_{D_i/\overline{X}}$. Let $\overline{D}_{(x)} \subseteq \overline{N}_{D_x/\overline{X}}$ denote the normal crossings divisor $D_0 \cup D_\infty$.

By the first section we have a functor

$$\mathrm{Isoc}_{uni}(\overline{X}_{log}/W) \to \mathrm{Isoc}_{uni}(x_{log}/W),$$

combining this with the canonical isomorphism $x_{log} \simeq 0_{log}$, in section 3 and applying the last lemma to obtain an equivalence of categories

$$\mathrm{Isoc}_{uni}(\overline{N}_{log}/W) \to \mathrm{Isoc}_{uni}(0_{log}/W),$$

we obtain a functor

$$\varphi^* : \mathrm{Isoc}_{uni}(\overline{X}_{log}/W) \to \mathrm{Isoc}_{uni}(\overline{N}_{log}/W).$$

Remark. Note that in the de Rham version of the above we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Mic}_{uni}(\overline{X}_{log}/K) & \longrightarrow & \mathrm{Mic}_{uni}(\overline{N}_{log}/K) \\ \downarrow & & \downarrow \\ \mathrm{Mic}_{reg}(X/K) & \longrightarrow & \mathrm{Mic}_{reg}(N/K). \end{array}$$

6.2. Description of φ^*. First we will give another description of the functor

$$\mathrm{Isoc}_{uni}(\overline{X}_{log}/W) \to \mathrm{Isoc}_{uni}(0_{log}/W).$$

We use the notation of section 2.(iii)., i.e. $\overline{\mathfrak{X}}$ is a lifting of $\overline{X}$ etc. Let $\psi : \mathcal{U}_{log} \to \overline{\mathfrak{X}}_{log,K}$, where $\mathcal{U} \subseteq (\mathfrak{N}_{\mathfrak{D}_{\mathfrak{x}}/\overline{\mathfrak{X}}})_K$ is a polydisc around zero, be a map such that $\psi(0) = \mathfrak{x}_K$; the map on the monoids is the identity map; the map induced by the differential at zero, $d_0(\psi)$ from $\oplus_{i\in I}(\mathfrak{N}_{\mathfrak{D}_i/\overline{\mathfrak{X}}})_K \simeq T_0(\mathfrak{N}_{\mathfrak{D}_{\mathfrak{x}}/\overline{\mathfrak{X}}})_K$ to $\oplus_{i\in I}(\mathfrak{N}_{\mathfrak{D}_i/\overline{\mathfrak{X}}})_K$ is the identity map; and ψ is an isomorphism onto its image. We have a map

$$\psi^* : \mathrm{Mic}_{uni}(\overline{\mathfrak{X}}_{K,log}/K) \to \mathrm{Mic}_{uni}(\mathcal{U}_{log}/K),$$

where $\mathcal{U}_{log}$ is the log scheme associated to the divisor obtained by pulling back $\mathfrak{D}_K$. Combining this with the restriction map, we obtain

$$\psi_0^* : \mathrm{Mic}_{uni}(\overline{\mathfrak{X}}_{K,log}/K) \to \mathrm{Mic}_{uni}(0_{\mathfrak{x},log}/K).$$

Note that here $0_{\mathfrak{x},log}$ denotes a log scheme over K and that in fact the category $\mathrm{Mic}_{uni}(0_{\mathfrak{x},log}/K)$ does not depend, up to canonical isomorphism, on the choices of the liftings since $\mathrm{Mic}(0_{\mathfrak{x},log}/K) \simeq \mathcal{T}_{\mathfrak{x}}/K$ and $\mathrm{Weil}^-(\mathcal{U}, \mathcal{D}_{\mathfrak{x}}) \simeq \mathrm{Weil}^-(\overline{X}, D_x)$.

Let $\mathcal{V} \subseteq (\mathfrak{N}_{\mathfrak{E}_{\mathfrak{y}}/\overline{\mathfrak{Y}}})_K$, and $\tilde{\psi} : \mathcal{V} \to \overline{\mathfrak{Y}}_K$ be another choice as above, with $\tilde{\psi}(0) = \mathfrak{y}_K$. Then we have a map

$$(\psi \times \tilde{\psi})^\sim : (\mathcal{U} \times \mathcal{V})^\sim_{log} \to (\overline{\mathfrak{X}}_K \times \overline{\mathfrak{Y}}_K)^\sim_{log}$$

induced by $\psi \times \tilde{\psi}$. The underlying map of schemes is the identity map on the exceptional divisors. Note that the exceptional divisors are respectively the products of the normal bundles of $\psi^*(\mathfrak{D}_{K,i}) \times \tilde{\psi}^*(\mathfrak{D}_{K,i})$ and $\mathfrak{D}_{K,i} \times \mathfrak{D}_{K,i}$ at $(0,0)$ and $(\mathfrak{x}_K, \mathfrak{y}_K)$. If $(E, \nabla) \in \mathrm{Isoc}_{uni}(\overline{X}_{log}/W)$ then by pulling back with $(\psi \times \tilde{\psi})^\sim$ we have a canonical isomorphism between the pullbacks of $\psi_0^*(E, \nabla)_{\overline{\mathfrak{X}}}$ and $\tilde{\psi}_0^*(E, \nabla)_{\overline{\mathfrak{Y}}}$ to the tube of the diagonal in $(0_{\mathfrak{x},log} \times 0_{\mathfrak{y},log})^\sim$. Here note that even though the diagonal is not defined, the tube of the diagonal is well-defined as the tube of the diagonal after $0_{\mathfrak{x},log}/W$ and $0_{\mathfrak{y},log}/W$ are identified by an isomorphism that induces the identity map on the special fibers. These isomorphisms on the tubes, viewed as log rigid analytic spaces, satisfy the cocycle condition and hence define an element of $\mathrm{Isoc}_{uni}(0_{log}/W)$ by lemma 5 and 6.

Lemma 7. *Let $(\mathfrak{N}, \mathfrak{D})/W$ and $(\mathfrak{M}, \mathfrak{E})/W$ be two formal vector bundles endowed with linear normal crossings divisors passing through 0, which are liftings of a vector bundle with normal crossings divisor $(N, D)/k$. Using the standard notation as above, let (E, ∇) and (F, ∇) be in $\mathrm{Mic}_{uni}(\overline{\mathfrak{N}}_{K,log}/K)$ and $\mathrm{Mic}_{uni}(\overline{\mathfrak{M}}_{K,log}/K)$. And let*

$$\alpha_0 : p_1^*(E, \nabla)_0 \to p_2^*(F, \nabla)_0$$

be an isomorphism between the pullbacks to the tube of the diagonal in $(0_{log} \times 0_{log})^\sim$ of the restrictions of (E, ∇) and (F, ∇) to 0_{log} in $\mathfrak{N}_K$ and $\mathfrak{M}_K$. Then there is a unique isomorphism

$$\alpha : p_1^*(E, \nabla) \to p_2^*(F, \nabla)$$

on the tube of the diagonal in $(\overline{\mathfrak{N}} \times \overline{\mathfrak{M}})^\sim_{K,log}$ that extends α_0.

Proof. First we will assume without loss of generality that the special fiber of the formal vector bundles is the trivial bundle k^r. In order to prove the lemma we will assume, without loss of generality by choosing bases whose reductions modulo p are the standard basis of k^r, that $(\mathfrak{N}, \mathfrak{D})$ and $(\mathfrak{M}, \mathfrak{E})$ are $\mathbb{A}^r$ endowed with the standard coordinate hyperplanes as the divisor. Furthermore we will assume by Lemma 3. that $(E, \nabla) = (F, \nabla) = (\mathcal{O}^r, d - \sum_{i \in I} N_i \, d\log z_i)$, for some nilpotent operators N_i. The existence follows from the fact that unipotent logarithmic connections on the generic fiber p-adically converge, that is induce logarithmic isocrystals. More explicitly the isomorphism from $p_1^*(E, \nabla)$ to $p_2^*(E, \nabla)$ on the tube of the diagonal is given by $\prod_{1 \le i \le r} e^{\log(1+t_i)N_i}$, where we let $\frac{z_i^{(2)}}{z_i^{(1)}} = 1 + t_i$, for $|t_i| < 1$. In particular note that $\alpha((\mathfrak{v}_K, \mathfrak{u}_K)) = \alpha_0([\mathfrak{v}_K, \mathfrak{u}_K])$, when $E = V_1 \otimes \mathcal{O}$ and $F = V_2 \otimes \mathcal{O}$ as in the statement of the lemma, where $\mathfrak{u}$ and $\mathfrak{v}$ have the same reduction $v \in N^*(k)$ modulo p, and $[\mathfrak{v}_K, \mathfrak{u}_K]$ denotes the point corresponding to $(\mathfrak{v}_K, \mathfrak{u}_K)$ in the exceptional divisor.

In order to see the uniqueness it is enough to show that for a unipotent vector bundle with logarithmic connection (E, ∇) on the tube $]\Delta_{\overline{N}}[_{(\overline{\mathfrak{N}} \times \overline{\mathfrak{M}})^\sim}$ of the diagonal in $(\overline{\mathfrak{N}} \times \overline{\mathfrak{M}})^\sim$ the restriction functor

$$\mathrm{H}^0_{dR}(]\Delta_{\overline{N}}[_{(\overline{\mathfrak{N}} \times \overline{\mathfrak{M}})^\sim}, (E, \nabla)) \to \mathrm{H}^0_{dR}(]0[_{(0_{log} \times 0_{log})^\sim}, (E, \nabla))$$

is injective. This follows from $\mathrm{H}^0_{dR}(]\Delta_{\overline{N}}[_{(\overline{\mathfrak{N}} \times \overline{\mathfrak{M}})^\sim}, (\mathcal{O}, d)) = K$ by induction on the nilpotence level of (E, ∇). □

Therefore we obtain a map

$$\mathrm{Isoc}_{uni}(\overline{X}_{log}/W) \to \mathrm{Isoc}_{uni}(0_{\mathfrak{x},log}/W).$$

In order to see that this is the same as the one constructed before we note that the map

$$(0_{\mathfrak{x},log} \times 0_{\mathfrak{y},log})^\sim \to (\mathcal{U} \times \mathcal{V})^\sim_{log} \to (\overline{\mathfrak{X}}_K \times \overline{\mathfrak{Y}}_K)^\sim_{log}$$

induced by ψ and $\tilde{\psi}$ factors as

$$(0_{\mathfrak{x},log} \times 0_{\mathfrak{y},log})^\sim \to (\mathfrak{x}_{K,log} \times \mathfrak{y}_{K,log})^\sim \to (\overline{\mathfrak{X}}_K \times \overline{\mathfrak{Y}}_K)^\sim_{log}.$$

The underlying schemes of $(0_{log} \times 0_{log})^\sim$ and $(\mathfrak{x}_{K,log} \times \mathfrak{y}_{K,log})^\sim$ are the exceptional divisors of $(\mathcal{U} \times \mathcal{V})^\sim_{log}$, and $(\overline{\mathfrak{X}}_K \times \overline{\mathfrak{Y}}_K)^\sim_{log}$ and the map induced between them is the identity map if they are identified via the isomorphisms of the form $\mathfrak{z}_{K,log} \simeq 0_{\mathfrak{z},log}$, $\mathfrak{z} \in \{\mathfrak{x}, \mathfrak{y}\}$ because of the condition on the derivative at 0 of the local isomorphisms ψ and $\tilde{\psi}$. This shows that the two functors from $\mathrm{Isoc}_{uni}(\overline{X}_{log}/W)$ to $\mathrm{Isoc}_{uni}(0_{\mathfrak{x},log}/W)$ are the same.

To give a description of φ^* we need to describe

$$\mathrm{Isoc}_{uni}(0_{log}/W) \to \mathrm{Isoc}_{uni}(\overline{N}_{log}/W).$$

Let $\mathfrak{N}_{\mathfrak{D}_{\mathfrak{x}}/\overline{\mathfrak{X}}}/W$ be a formal vector bundle, lifting $N_{D_x/\overline{X}}$ endowed with a linear normal crossings divisor at 0. We do not assume that the lifting in fact comes from the tangent space of a lifting $\overline{\mathfrak{X}}$ of $\overline{X}$ but continue to use that notation for consistency. The functor, defined only up to canonical isomorphism,

$$e_{\overline{\mathfrak{X}}} : \mathcal{T}_{x,uni}/K \simeq \mathrm{Mic}_{uni}(0_{\mathfrak{x},log}/K) \to \mathrm{Mic}_{uni}((\overline{\mathfrak{N}}_{\mathfrak{D}_{\mathfrak{x}}/\overline{\mathfrak{X}}})_{K,log}/K)$$

is the one that sends $(V, \{N_i\}_{i \in I})$ to $(V \otimes_K \mathcal{O}_{\overline{\mathfrak{N}}}, d - \sum_{i \in I} N_i d\log z_i)$, where $\{z_i\}_{i \in I}$ is a linear system of coordinates for $\mathfrak{N}_K$ such that the divisor on $\mathfrak{N}_K$ is defined by $\prod_{i \in I} z_i = 0$. Note that since the z_i are defined only up to a scalar multiple the $d\log z_i$ are well-defined.

Let $\mathfrak{N}_{\mathfrak{E}_{\mathfrak{y}}/\overline{\mathfrak{Y}}}/W$ be another such lifting. Then for $(E, \nabla) \in \mathrm{Isoc}_{uni}(0_{log}/W)$, by the construction we have

$$e_{\overline{\mathfrak{X}}}(E, \nabla) \in \mathrm{Mic}_{uni}((\overline{\mathfrak{N}}_{\mathfrak{D}_{\mathfrak{x}}/\overline{\mathfrak{X}}})_{K,log}/K) \ \text{ and } \ e_{\overline{\mathfrak{Y}}}(E, \nabla) \in \mathrm{Mic}_{uni}((\overline{\mathfrak{N}}_{\mathfrak{E}_{\mathfrak{y}}/\overline{\mathfrak{Y}}})_{K,log}/K)$$

and an isomorphism

$$\alpha_0 : p_1^* e_{\overline{\mathfrak{X}}}(E, \nabla)_0 \to p_2^* e_{\overline{\mathfrak{Y}}}(E, \nabla)_0.$$

And hence an isomorphism

$$\alpha : p_1^* e_{\overline{\mathfrak{X}}}(E, \nabla) \to p_2^* e_{\overline{\mathfrak{Y}}}(E, \nabla)$$

by the last lemma. Since the isomorphisms on the log points satisfy the cocycle condition, the isomorphisms for the connections on the $\overline{\mathfrak{N}}'_K$s also satisfy the cocycle conditon by the uniqueness statement in the last lemma. Therefore the data of objects $e_{\overline{\mathfrak{X}}}(E,\nabla)$, for each lifting $\overline{\mathfrak{X}}$ as above, together with the isomorphisms of their pullbacks to the blow-up of the products for different liftings define an element $e(E,\nabla) \in \mathrm{Isoc}_{uni}(\overline{N}_{log}/W)$. This is a more explicit description of the inverse of

$$\mathrm{Isoc}_{uni}(\overline{N}_{log}/W) \to \mathrm{Isoc}_{uni}(0_{log}/W)$$

that will be useful below.

6.3. Comparison with ordinary basepoints. Let Y/k be a smooth variety, $y \in Y(k)$, $i : y \to Y$ the inclusion. Then we denote the fiber functor $i^* : \mathrm{Isoc}(Y/W) \to Vec_K$ by $\omega(y)$.

Choosing $v \in N^*_{D_x/\overline{X}}(k)$, and composing φ^* with

$$\omega(v) : \mathrm{Isoc}_{uni}(\overline{N}_{log}/W) \to \mathrm{Isoc}_{uni}(N^*/W) \to Vec_K$$

we obtain the fiber functor $\varphi_v^* : \mathrm{Isoc}_{uni}(\overline{X}_{log}/W) \to Vec_K$.

Proposition 1. *There is a canonical natural isomorphism $\varphi_v^* \simeq fib(v)$. Here $fib(v)$ is the tangential basepoint defined above.*

Proof. Let $(E,\nabla) \in \mathrm{Isoc}_{uni}(\overline{X}_{log}/W)$ and let $\overline{\mathfrak{X}}$, etc. be a lifting, and let $\psi : \mathcal{U} \to \overline{\mathfrak{X}}_K$ be a map as above. Then by the description of φ^* above,

$$(\varphi^*(E,\nabla))_{\overline{\mathfrak{X}},\varphi} = (E_{\overline{\mathfrak{X}}}(\mathfrak{x}_K) \otimes_K \mathcal{O}_{\overline{\mathfrak{N}}_K}, d - \sum_{i\in I} \mathrm{res}_{\mathfrak{D}_{K,i}}(E,\nabla)_{\overline{\mathfrak{X}}}(\mathfrak{x}_K) d\log z_i)$$

since $\mathrm{res}_{\varphi^{-1}(\mathfrak{D}_{K,i})}\varphi^{-1}(E,\nabla)_{\overline{\mathfrak{X}}}(\mathfrak{x}_K) = \mathrm{res}_{\mathfrak{D}_{K,i}}(E,\nabla)_{\overline{\mathfrak{X}}}(0)$. Together with the choices above let $\mathfrak{v} \in \mathfrak{N}^*_{\mathfrak{D}_{\mathfrak{x}}/\overline{\mathfrak{X}}}(W)$ lifting v. Then by the above formula, the realization $\varphi_v^*(\overline{\mathfrak{X}},\psi,\mathfrak{v})$ of φ_v^* is the map sending (E,∇) as above to $(E_{\overline{\mathfrak{X}}}(\mathfrak{x}_K) \otimes_K \mathcal{O}_{\overline{\mathfrak{N}}_K})(\mathfrak{v}_K) = E_{\overline{\mathfrak{X}}}(\mathfrak{x}_K)$. Similarly by the description given in 1.(iii). $fib(v)_{\overline{\mathfrak{X}},\mathfrak{v}}(E,\nabla) = E_{\overline{\mathfrak{X}}}(\mathfrak{x}_K)$.

Let $(\overline{\mathfrak{Y}},\tilde{\psi},\mathfrak{u})$ be another such data of a lifting. Then the isomorphism between $fib_{\overline{\mathfrak{X}},\mathfrak{v}}(E,\nabla)$ and $fib_{\overline{\mathfrak{Y}},\mathfrak{u}}(E,\nabla)$ is the one obtained from pulling the connections to the point $[\mathfrak{v}_K,\mathfrak{u}_K] \in (\overline{\mathfrak{X}}_K \times \overline{\mathfrak{Y}}_K)^\sim$. Similarly the isomorphism between $\varphi_v^*(\overline{\mathfrak{X}},\psi,\mathfrak{v})(E,\nabla)$ and $\varphi_v^*(\overline{\mathfrak{Y}},\tilde{\psi},\mathfrak{u})(E,\nabla)$ is the one obtained by pulling back the connections to the point $(\mathfrak{v}_K,\mathfrak{u}_K) \in (\overline{\mathfrak{N}}_{\overline{\mathfrak{X}}} \times \overline{\mathfrak{N}}_{\overline{\mathfrak{Y}}})^\sim_K$. Therefore in order to prove the claim we only need to see that for $(V_1 \otimes \mathcal{O},\nabla) \in \mathrm{Mic}_{uni}((\overline{\mathfrak{N}}_{\overline{\mathfrak{X}}})_{K,log}/K)$ and $(V_2 \otimes \mathcal{O},\nabla) \in \mathrm{Mic}_{uni}((\overline{\mathfrak{N}}_{\overline{\mathfrak{Y}}})_{K,log}/K)$ and an isomorphism between their pullbacks to $(\overline{\mathfrak{N}}_{\overline{\mathfrak{X}}} \times \overline{\mathfrak{N}}_{\overline{\mathfrak{Y}}})^\sim_K$, the isomorphism at the point $[\mathfrak{v},\mathfrak{u}]$ is the same as the one at $(\mathfrak{v},\mathfrak{u})$. But this was shown in the proof of Lemma 4. □

6.4. Tangential basepoints for unipotent overconvergent isocrystals. We continue to use the notation above, and let $\mathrm{Isoc}^\dagger_{\overline{X}}(X/W)$ denote the category of isocrystals overconvergent along $D \subseteq \overline{X}$. Then the natural map

$$\mathrm{Isoc}_{uni}(\overline{X}_{log}/W) \to \mathrm{Isoc}^\dagger_{\overline{X},uni}(X/W)$$

is an equivalence of categories. This follows, in the usual way, from the fact that for any log F^k-isocystal $\mathcal{E}$, in particular $\mathcal{O}_{\overline{X}} \otimes K$, on $\overline{X}_{log}$ the natural map

$$\mathrm{H}^i_{rig}(\overline{X}_{log}/W,\mathcal{E}) \to \mathrm{H}^i_{rig,\overline{X}}(X/W, j^\dagger\mathcal{E})$$

is an isomorphism ([Shi], proof of (2.4.1)) for all i, where $j : X \hookrightarrow \overline{X}$ is the inclusion. Therefore by the above construction we obtain a fiber functor on $\mathrm{Isoc}^\dagger_{\overline{X},uni}(X/W)$, and hence a fiber functor

$$fib(v) : \mathrm{Isoc}^\dagger_{uni}(X/W) \to Vec_K,$$

for $v \in N^*_{D_x/\overline{X}}(k)$.

7. Drinfel'd-Ihara relation

7.1. de Rham basepoint. From now on let $X := \mathbb{P}^1 \setminus \{0, 1, \infty\}$, and $\overline{X} := \mathbb{P}^1$. In the following when we write $\mathbb{P}^1/\mathbb{Q}$ we will always assume that it is endowed with a specific choice of a coordinate function, i.e. a rational function $z \in k(\mathbb{P}^1/\mathbb{Q})$ such that $\mathbb{Q}(z) = k(\mathbb{P}^1/\mathbb{Q})$, where $k(\mathbb{P}^1/\mathbb{Q})$ denotes the field of rational functions on $\mathbb{P}^1/\mathbb{Q}$.

Let

$$M_{0,n} := \{(x_0, \cdots, x_{n-1}) \in (\mathbb{P}^1)^n | \text{ all } x_i \text{ are different}\}/PGL_2,$$

where PGL_2 acts diagonally by linear fractional transformations.

Lemma 8. *For $n \geq 4$, $M_{0,n}$ has a compactification $\overline{M}_{0,n}$ with $\overline{M}_{0,n} \backslash M_{0,n}$ a simple normal crossings divisor and $H^1_B((\overline{M}_{0,n})_{\mathbb{C}}, \mathbb{Z}) = 0$.*

Proof. First note that by using a linear fractional transformation that sends (x_0, x_1, x_2) to $(0, 1, \infty)$ we can identify $M_{0,n}$ with $X^{n-3} \setminus \Delta_h$, where

$$\Delta_h := \{(x_1, \cdots, x_{n-3}) | x_i \neq x_j \text{ for } i \neq j\}$$

is the hyperdiagonal in X^{n-3}. A compactification $\overline{M}_{0,n}$ of $M_{0,n} \subseteq (\mathbb{P}^1)^{n-3}$ with $\overline{M}_{0,n} \setminus M_{0,n}$ a simple normal crossings divisor is obtained by a succession of blowings up of $(\mathbb{P}^1)^{n-3}$ along linear subvarieties. Since $(\mathbb{P}^1)^{n-3}$ is simply connected and the blowings up do not change the first cohomology group we have $H^1_B((\overline{M}_{0,n})_{\mathbb{C}}, \mathbb{Z}) = 0$. □

The above lemma together with Grothedieck's comparison theorem gives

$$H^1_{dR}((\overline{M}_{0,n})_{\mathbb{C}}, (\mathcal{O}, d)) = 0,$$

and hence $H^1(\overline{M}_{0,n}, \mathcal{O}) = 0$. Therefore one has a canonical fiber functor on

$$\mathrm{Mic}_{uni}(M_{0,n}/\mathbb{Q})$$

defined in [De]. This can be described as follows. For any $(E, \nabla) \in \mathrm{Mic}_{uni}(M_{0,n}/\mathbb{Q})$, the underlying vector bundle $\overline{E}$ of the canonical extension $(\overline{E}, \nabla) \in \mathrm{Mic}_{uni}(\overline{M}_{0,n}, \log D)$ is trivial [De], where $D := \overline{M}_{0,n} \setminus M_{0,n}$. Therefore the canonical map

$$\Gamma(\overline{M}_{0,n}, \overline{E}) \otimes \mathcal{O}_{\overline{M}_{0,n}} \simeq \overline{E}$$

is an isomorphism and the functor

$$\omega(dR) \, (:= \Gamma(\overline{M}_{0,n}, \cdot)) : \mathrm{Mic}_{uni}(M_{0,n}/\mathbb{Q}) \to \mathrm{Vec}_{\mathbb{Q}}$$

is a fiber functor.

This a priori depends on the choice of a compactification of $M_{0,n}$ with zero first Betti cohomology. Let $\overline{M}^1_{0,n}$ and $\overline{M}^2_{0,n}$ be two such compactifications. By applying Hironaka's resolution of singularities to the closure $\Delta(M_{0,n})^-$ of the image of $M_{0,n}$ in $\overline{M}^1_{0,n} \times \overline{M}^2_{0,n}$ under the diagonal map we obtain a compactification $\overline{M}^{12}_{0,n}$ of $M_{0,n}$ with the complement a simple normal crossings divisor together with maps $\overline{M}^{12}_{0,n} \to \overline{M}^1_{0,n}$ and $\overline{M}^{12}_{0,n} \to \overline{M}^2_{0,n}$ that commute with the inclusions of $M_{0,n}$. Let $(E, \nabla) \in \mathrm{Mic}_{uni}(M_{0,n}/\mathbb{Q})$, and $(\overline{E}^i, \nabla)$ its canonical extension to $\overline{M}^i_{0,n}$, for $i \in \{1, 2\}$. Then since $\mathrm{supp}(\pi_i^*(D_i)) \subseteq D_{12}$ we have $\pi_i^*(\overline{E}^i \nabla) \in \mathrm{Mic}(\overline{M}^{12}_{0,n}(\log D_{12})/\mathbb{Q})$. Furthermore since the exponents of the pull-backs are linear combinations of the original exponents, which are zero by the definition of canonical extension, the exponents of the pull-backs are zero as well. And hence $\pi_i^*(\overline{E}^i \nabla) \in \mathrm{Mic}_{uni}(\overline{M}^{12}_{0,n}(\log D_{12})/\mathbb{Q})$. Therefore the identification $\pi_1^*(\overline{E}^1 \nabla)|_{M_{0,n}} = \pi_2^*(\overline{E}^2 \nabla)$ extends to $\overline{M}^{12}_{0,n}$ and after taking global sections induces an isomorphism $\Gamma(\overline{M}^{12}_{0,n}, \overline{E}^1) \simeq \Gamma(\overline{M}^{12}_{0,n}, \overline{E}^2)$. A similar argument shows that the cocycle condition is satisfied for three different compactifications and hence the definition of the canonical de Rham fiber functor is independent of the compactification satisfying the properties above, and is functorial with respect to arbitrary maps between $M_{0,n}$.

If x is a (tangential) basepoint of $M_{0,n}$ then the natural maps $\Gamma(\overline{M}_{0,n}, \overline{E}) \to \overline{E}(x)$ induce an isomorphism of the fiber functors $\omega(dR)$ and $\omega(x)$. And hence there is a canonical path between the

fiber functors $\omega(x)$ and $\omega(y)$ in the de Rham theory; this is denoted by ${}_ye(dR)_x$. As above this path is independent of the compactification in the case where x and y are ordinary points.

7.2. Fundamental group of $M_{0,n}$. Let $\pi_{1,dR}(M_{0,n},\omega(dR))$ denote the fundamental group of the tannakian category $\mathrm{Mic}_{uni}(M_{0,n}/\mathbb{Q})$ at the fiber functor $\omega(dR)$; this is a pro-unipotent algebraic group.

Lemma 9. *The natural map* $p_n : M_{0,n} \to M_{0,n-1}$ *with*

$$p_n([x_0,\cdots,x_{n-1}]) = [x_0,\cdots,x_{n-2}]$$

induces an exact sequence

$$0 \to \pi_{1,dR}(F_x,\omega(dR)) \to \pi_{1,dR}(M_{0,n},\omega(dR)) \to \pi_{1,dR}(M_{0,n-1},\omega(dR)) \to 0$$

where F_x *is the fiber of* p_n *containing* $x := [x_0,\cdots,x_{n-1}]$.

Proof. First assume that the basefield is $\mathbb{C}$. Then p_n is a locally trivial fibration with fibers isomorphic to $\mathbb{P}^1$ minus $n-1$ points. Hence we get a homotopy exact sequence for the topological fundamental groups

$$\cdots \to \pi_2(M_{0,n-1},p_n(x)) \to \pi_1(F_x,x) \to \pi_1(M_{0,n},x) \to \pi_1(M_{0,n-1},p_n(x)) \to \cdots.$$

Since F_x is connected we obtain the exact sequence

$$\pi_1(F_x,x) \to \pi_1(M_{0,n},x) \to \pi_1(M_{0,n-1},p_n(x)) \to 0$$

Let G be an abstract group, and $\{Z^i(G)\}_{1\le i}$ the central descending series, i.e. $Z^1(G) := G$ and $Z^{i+1}(G) := [G,Z^i(G)]$. Then $G^{[N]} := (G/Z^{N+1}(G))/tors$ (note that since $G/Z^{N+1}(G)$ is nilpotent the set of its torsion elements form a subgroup [Ba]) is canonically imbedded into $G^{[N]\,*}$, the universal nilpotent torsion free divisible group receiving a map from $G^{[N]}$. If $g \in G^{[N]\,*}$ then there is an $n \in \mathbb{Z}_{>0}$ such that $g^n \in G^{[N]}$, and for every $g \in G^{[N]}$ and $n \in \mathbb{Z}_{\ge 0}$ there exists a unique $h \in G^{[N]\,*}$ such that $h^n = g$ ([Ba], section 8.3). These imply that

$$\pi_1(F_x,x)^{[N]\,*} \to \pi_1(M_{0,n},x)^{[N]\,*} \to \pi_1(M_{0,n-1},p_n(x))^{[N]\,*} \to 0$$

is exact.

For a pro-unipotent algebraic group H, let $H^{(N)}$ denote its largest quotient of nilpotence level $\le N$. Since, by the Riemann-Hilbert correspondence, $\pi_{1,dR}(\cdot)^{(N)}$ is the unipotent algebraic envelope of $\pi_1(\cdot)^{[N]*}$ over $\mathbb{C}$, the exact sequence above implies that

$$\pi_{1,dR}(F_x,x)^{(N)} \to \pi_{1,dR}(M_{0,n},x)^{(N)} \to \pi_{1,dR}(M_{0,n-1},p_n(x))^{(N)} \to 0$$

is exact. Since any unipotent integrable connection on F_x of level N can be extended to a unipotent integrable connection of level N on $M_{0,n}$ the first map is injective. By taking the inverse limits we obtain the exact sequence

$$0 \to \pi_{1,dR}(F_x,x) \to \pi_{1,dR}(M_{0,n},x) \to \pi_{1,dR}(M_{0,n-1},p_n(x)) \to 0$$

since the inverse system $\{\pi_{1,dR}(.\,,.)^{(N)}\}_{N\ge 1}$ satisfies the Mittag-Leffler condition. This gives the statement in the lemma when the basefield is $\mathbb{C}$. To obtain it in the case when the base field is $\mathbb{Q}$ we note that the unipotent de Rham fundamental group commutes with the base change of the fields [De]. □

Remark. Note that the sequence

$$0 \to \pi_1(F_x,x) \to \pi_1(M_{0,n},x) \to \pi_1(M_{0,n-1},p_n(x)) \to 0$$

is exact on the left as well. This follows from the fact that $\pi_i(M_{0,n},\ .) = 0$ for $i \ge 2$ and $n \ge 4$. To see this first note that the fibers $F^{(n)}$ of the maps p_n, being isomorphic to $\mathbb{P}^1$ minus n-1 points have the unit disc as a covering space (by uniformization theory), therefore $\pi_i(F^{(n)},\ .) = 0$ for $i \ge 2$. Then the claim follows by induction from this and the homotopy sequences for the fibrations p_n.

Residues. Let

$$M^*_{0,n} := \{(x_0,\cdots,x_{n-1}) \in (\mathbb{P}^1)^n \mid \text{at most two of the } x_i \text{ are equal}\}/PGL_2.$$

We have $M_{0,n} \subseteq M^*_{0,n}$ as an open subvariety with the complement a simple normal crossings divisor. Let $D_{ij} \subseteq M^*_{0,n}$ denote the divisor defined by the image of $x_i = x_j$. By the above lemma, using induction, we see that $H_{1,B}(M_{0,n})$ is generated by the loops around the divisors D_{ij}. Therefore the image of $H_{1,B}(M_{0,n})$ in $H_{1,B}(M^*_{0,n})$ is zero. Since $H_{1,B}(M^*_{0,n}, M_{0,n}) = 0$, by using Mayer-Vietoris sequence and excision for a suitable cover, we see that $H_{1,B}(M^*_{0,n}) = 0$. Therefore we may choose $\overline{M}_{0,n}$ as in the previous lemma such that $M^*_{0,n} = \overline{M}_{0,n} \setminus D_0$, where $D_0 \subseteq \overline{M}_{0,n}$ is a divisor contained in $\overline{M}_{0,n} \setminus M_{0,n}$.

Let $\overline{D}_{ij}$ denote the closure of D_{ij} in $\overline{M}_{0,n}$. Let $x \in \overline{D}_{ij}(\overline{\mathbb{Q}})$ and $\{t_k\}_{1\leq k\leq n-3}$ a system of parameters on $\overline{M}_{0,n}$ at x such that the divisor $\overline{M}_{0,n} \setminus M_{0,n}$ is defined by $t_1 \cdots t_r = 0$ at x and $\overline{D}_{ij}$ is defined by $t_1 = 0$ at x. If we let $(\overline{E}, \nabla)$ denote the canonical extension of (E, ∇) we obtain a map

$$-\nabla_{t_1 \frac{\partial}{\partial t_1}}(x) : \overline{E}(x) \to \overline{E}(x),$$

which is independent of the choice of the system of parameters as above. As usual we denote this map by $\mathrm{res}_{x,\overline{D}_{ij}}(\overline{E}, \nabla)$ This map satisfies

$$\mathrm{res}_{x,\overline{D}_{ij}}((\overline{E}, \nabla) \otimes (\overline{F}, \nabla)) = id_{\overline{E}(x)} \otimes \mathrm{res}_{x,\overline{D}_{ij}}((\overline{F}, \nabla) + \mathrm{res}_{x,\overline{D}_{ij}}((\overline{E}, \nabla) \otimes id_{\overline{F}(x)},$$

and hence gives an element $\mathrm{res}_{x,\overline{D}_{ij}} \in \mathrm{Lie}\pi_{1,dR}(M_{0,n}, x)$. Using the canonical isomorphism

$$\Gamma((\overline{M}_{0,n})_{\overline{\mathbb{Q}}}, \overline{E}) \simeq \overline{E}(x)$$

we obtain an algebraic map

$$(\overline{D}_{ij})_{\overline{\mathbb{Q}}} \to \mathrm{End}(\Gamma(\overline{M}_{0,n})_{\overline{\mathbb{Q}}}, \overline{E}).$$

Since $(\overline{D}_{ij})_{\overline{\mathbb{Q}}}$ is proper and integral this map is in fact constant. And by descent this is in fact defined over $\mathbb{Q}$ and we obtain elements $\mathrm{res}_{ij} \in \mathrm{Lie}\pi_{1,dR}(M_{0,n}, \omega(dR))$.

If $\{i, j\} \cap \{k, l\} = \emptyset$ then $D_{ij} \cap D_{kl} \neq \emptyset$ and computing the residues at a point in $D_{ij} \cap D_{kl}$ we see that $[\mathrm{res}_{ij}, \mathrm{res}_{kl}] = 0$ by the integrability of the connection. Similarly by computing the residues along a fiber of the projection $M_{0,n} \to M_{0,n-1}$ that maps $[x_0, \cdots, x_j, \cdots, x_{n-1}]$ to $[x_0, \cdots, \hat{x}_j, \cdots, x_{x_n-1}]$, we see that $\sum_i \mathrm{res}_{ij} = 0$, where we let $\mathrm{res}_{ii} = 0$ by convention. Let

$$H_n := \mathrm{Lie} << e_{ij} >>_{0\leq i,j\leq n-1} /(e_{ii}, e_{ij} - e_{ji}, \sum_i e_{ij}, [e_{ij}, e_{kl}]| \text{ for } \{i, j\} \cap \{k, l\} = \emptyset),$$

with Lie $<< \cdot >>$ denoting the free pro-nilpotent Lie algebra generated by the arguments. We have a map $H_n \to \mathrm{Lie}\pi_{1,dR}(M_{0,n}, \omega(dR))$. This map is surjective since for a space Y that has a compactification $\overline{Y}$ as above with $H^1_B(\overline{Y}_{\mathbb{C}}) = 0$, $\mathrm{Lie}\pi_{1,dR}(Y, \omega(dR))$ is generated by the dual of $H^1_{dR}(Y/\mathbb{Q})$ [De].

Corollary 1. *The exact sequence in the statement of the previous lemma has a natural splitting and gives*

$$\pi_{1,dR}(M_{0,n}, \omega(dR)) \simeq \pi_{1,dR}(M_{0,n-1}, \omega(dR)) \ltimes \pi_{1,dR}(F^{(n)}, \omega(dR)),$$

where $F^{(n)} \simeq \mathbb{P}^1 \setminus \{a_1, \cdots, a_{n-1}\}$ for some dictinct points a_i.

Proof. Let

$$H_n := \mathrm{Lie} << e_{ij} >>_{1\leq i,j\leq n} /(e_{ii},\, e_{ij} - e_{ji},\, \sum_j e_{ij},\, [e_{ij}, e_{kl}]| \text{ for } \{i, j\} \cap \{k, l\} = \emptyset).$$

There are exact sequences

$$0 \to G_n \to H_n \to H_{n-1} \to 0$$

where $G_n := Lie << e_{in} >> /(\sum_i e_{in})$

And there is a natural splitting $H_{n-1} \to H_n$ mapping e_{ij} to e_{ij}.

In other words

$$H_n \simeq H_{n-1} \ltimes G_n$$

where the action of H_{n-1} on G_n is determined by

$$[e_{ij}, e_{kn}] = 0 \text{ if } \{i, j\} \cap \{k, n\} = \emptyset$$

$$\text{and } [e_{ij}, e_{in}] \;=\; -[\sum_{k\neq i} e_{kj}, e_{in}] = -[e_{jn}, e_{in}] \text{ if } 1 \leq i, j \leq n-1.$$

As we have seen above there are surjections

$$H_n \to \mathrm{Lie}\pi_{1,dR}(M_{0,n}, \omega(dR)).$$

These fit into commutative diagrams

$$\begin{array}{ccccccccc} 0\to & G_n & \to & H_n & \to & H_{n-1} & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0\to & \mathrm{Lie}\pi_{1,dR}(F^{(n)}, \omega(dR)) & \to & \mathrm{Lie}\pi_{1,dR}(M_{0,n}, \omega(dR)) & \to & \mathrm{Lie}\pi_{1,dR}(M_{0,n-1}, \omega(dR)) & \to 0 \end{array}$$

with exact rows and this implies that they are in fact isomorphisms

$$H_n \simeq \mathrm{Lie}\pi_{1,dR}(M_{0,n}, \omega(dR))$$

by induction. □

From now on we will write e_{ij} instead of res_{ij}.

7.3. Basepoints on $M_{0,4}$ and $M_{0,5}$. For $i, j \in \{0, 1, \infty\}$ let t_{ij} denote the unit tangent vector at the point i that points in the direction from i to j. For example, $t_{01} := \frac{d}{dz}$ at 0, $t_{10} := -\frac{d}{dz}$ at 1, $t_{\infty 0} := z^2 \frac{d}{dz}$ at ∞ etc. Note that $X \simeq M_{0,4}$ by the map $z \to [0, z, 1, \infty]$, and $\mathbb{P}^1$ can be viewed as the compactification

$$\{(x_0, x_1, x_2, x_3) | x_0 \neq x_2,\ x_0 \neq x_3,\ x_2 \neq x_3\}/PGL_2$$

of $M_{0,4}$. Therefore we may view t_{ij} as basepoints on $M_{0,4}$.

Note that $M^*_{0,5}$ is a compactification of $M_{0,5}$. We will use the following tangential basepoints on $M^*_{0,5}$ at the points

$$[\{x_0, x_1\}, \{x_2, x_3\}, x_4],\ [\{x_0, x_1\}, x_2, \{x_3, x_4\}],\ [x_0, \{x_1, x_2\}, \{x_3, x_4\}],$$
$$[x_0\}, \{x_1, x_2\}, x_3, \{x_4],\ [x_0\}, x_1, \{x_2, x_3\}, \{x_4],$$

where by, say $[\{x_0, x_1\}, \{x_2, x_3\}, x_4]$ we mean the point $[x_0, x_1, x_2, x_3, x_4] \in \overline{M}_{0,5}$ with $x_0 = x_1$ and $x_2 = x_3$, and we let $t_{01,23}$ denote a tangent vector at that point that maps to the previously defined tangent vectors on X under the map that sends

$$[x_0, x_1, x_2, x_3, x_4] \to [x_0, x_2, x_3, x_4]$$

and the map that sends

$$[x_0, x_1, x_2, x_3, x_4] \to [x_0, x_1, x_2, x_4].$$

There are four different choices, however in the crystalline setting the choice between these four points will not be important; see the lemma below. Similarly we choose tangent vectors at the four remaining basepoints with the same property.

7.4. Frobenius. Let $Y/\mathbb{Q}_p$ be a smooth variety, and assume that there is a proper, smooth model $\overline{\mathfrak{Y}}/\mathbb{Z}_p$ and a simple relative normal crossings divisor $\mathfrak{D} \subseteq \overline{\mathfrak{Y}}$ whose irreducible components are defined over $\mathbb{Z}_p$, with a fixed isomorphism $\mathfrak{Y}_{\mathbb{Q}_p} \simeq Y$, where $\mathfrak{Y} := \overline{\mathfrak{Y}} \setminus \mathfrak{D}$. Using the isomorphism $\mathrm{Mic}_{uni}(Y/\mathbb{Q}_p) \simeq \mathrm{Isoc}^{\dagger}_{uni}(\mathfrak{Y} \otimes \mathbb{F}_p/\mathbb{Z}_p)$ we obtain a frobenius action $F^* : \mathrm{Mic}_{uni}(Y/\mathbb{Q}_p) \to \mathrm{Mic}_{uni}(Y/\mathbb{Q}_p)$ defined in [De] (see also [CS] and [Ün]). And choosing (tangential) basepoints x and y with finite reduction we obtain a map $F_* : {}_y\mathcal{G}_{dR,x}(Y/\mathbb{Q}_p) \to {}_y\mathcal{G}_{dR,x}(Y/\mathbb{Q}_p)$, where by $\mathcal{G}_{dR}$ we denote the de Rham fundamental groupoid. In fact the frobenius is independent of the choice of the model, but we will not need this below.

p-adic integration. Let $\mathbb{Q}_{p,st}$ denote the ring of polynomials $\mathbb{Q}_p[l(p)]$, where $l(p)$ is a formal variable that could be thought of as (a multi-valued) $\log p$. Then $\log z : D(1, 1^-) \to \mathbb{Q}_p$ uniquely extends to an additive map $\log z : \mathbb{Q}_p^* \to \mathbb{Q}_{p,st}$ such that $\log p = l(p)$.

Let $Y/\mathbb{Q}_p$ be a variety with a model $\overline{\mathfrak{Y}}/\mathbb{Z}_p$ etc. as above and $x, y \in X(\mathbb{Q}_p)$. Then Vologodsky [Vo] extending the work of Coleman, Colmez, Besser, shows that there is a canonical path ${}_yc_x \in {}_y\mathcal{G}_{dR,x}(Y/\mathbb{Q}_p)(\mathbb{Q}_{p,st})$ such that $F_*({}_yc_x) = {}_yc_x$, and ${}_yc_x$ has image ${}_y1_x$ under the canonical projection

of ${}_y\mathcal{G}_{dR,x}(\mathbb{Q}_{p,st})$ to $\mathbb{Q}_{p,st}$. F coincides with the above when x and y have finite reduction with respect to the given model; in fact in this case the element ${}_yc_x$ is defined over $\mathbb{Q}_p \subseteq \mathbb{Q}_{p,st}$.

Similarly there is a canonical path ${}_bc_a$ even when a and b are tangential basepoints. The same proof as in loc. cit. p. 17 extends to this case to show the existence and uniqueness of the path satisfying the properties above.

In order to describe this path, without loss of generality by the compatibility with respect to concatenation, we will describe ${}_uc_x$ where x is a genuine point and $u \in N^*_{D_y/\overline{Y}}(\mathbb{Q}_p)$, which is not necessarily of finite reduction with respect to the given model.

Let $\overline{Y}_{an}$ be the associated analytic space, $\mathcal{U} \subseteq N_{D_y/\overline{Y}}$ a polydisc around zero and $\varphi : \mathcal{U}_{log} \to \overline{Y}_{log,an}$ be a map such that $\varphi(0) = y$; the map on the monoids is the identity map; the map induced by the differential at zero is the identity map; and φ is an closed immersion. Assume without loss of generality that x is in the image of φ and u is in $\mathcal{U}$. Then we claim that the canonical crystalline path between $\omega(x) \otimes \mathbb{Q}_{p,st}$ and $\omega(u) \otimes \mathbb{Q}_{p,st}$ is given by $\lim_{\epsilon\to 0}({}_uc_\epsilon \cdot {}_{\varphi(\epsilon)}c_x)$ where ${}_uc_\epsilon$ is a path on $N_{D_y/\overline{Y}} \setminus D_{(y)}$, ${}_{\varphi(\epsilon)}c_x$ is a path on Y and $\omega(\epsilon)$ and $\omega(\varphi(\epsilon))$ are identified by the fact that the restriction to $\mathcal{U}$ of the pull-back to the tangent space of a unipotent connection is its pull-back via φ as described above.

First in order to see that the above limit exists by choosing coordinates and choosing a local trivialization $(\mathcal{O}^m, d - \sum_{1\le i\le r} N_i \frac{dz_i}{z_i})$ where N_i are nilpotent, of the the underlying bundle of a unipotent vector bundle with connection, we are reduced to showing that: if

$$\varphi : D(0,1^-)^r_{log} \to D(0,1^-)^n_{log}$$

is a closed immersion of logarithmic analytic spaces, where both of the spaces are endowed with the log structure associated to $z_1 \cdots z_r = 0$, such that $\varphi(0) = 0$; and $\frac{\varphi(z_i)}{z_i} \in 1 + \mathfrak{m}_{\overline{Y},0}$ then

$$\lim_{\epsilon\to 0} \prod_i exp(N_i \log \frac{u_i}{\epsilon_i}) \cdot \prod_i exp(N_i \log \frac{\varphi(\epsilon)_i}{x_i})$$

exists. Here log denotes the multivalued extension of the logarithm described above. Since

$$\lim_{\epsilon\to 0} \log \frac{\varphi(\epsilon)_i}{\epsilon_i} = 0$$

the above limit exists and is equal to $\prod_i exp(N_i \log \frac{u_i}{x_i})$. The standard arguments as in the section on tangential basepoints show that the definition does not depend on the choice of φ. Let $\overline{c}$ denote this path just described. The above argument also shows that if (E, ∇) is a unipotent log isocrystal on $\overline{Y}_{log}$ with a local trivialization $(\mathcal{O}^m, d - \sum N_i \frac{dz_i}{z_i})$ and φ is a map with the properties as above from an open disc in $N_{D_y/\overline{Y}}$ to $\overline{Y}$ then

$$\lim_{v\to 0} \prod_i (exp(-N_i \log \frac{u_i}{v_i})\, {}_u\overline{c}_{\varphi(v)}) = 1.$$

In order to see that ${}_u\overline{c}_x$ is the canonical crystalline path we need to show the invariance under frobenius. Since the path is invariant under frobenius between genuine basepoints to prove the invariance in general it suffices to prove this invariance for the limit. We choose local coordinates and a local trivialization of (E, ∇) as above.

Let $\mathcal{F}$ be a local lifting of the frobenius on the special fiber to a neighborhood of y in $\overline{Y}_{log}$ relative to the model $\overline{\mathfrak{Y}}$, that fixes y. Then the principal part $P(\mathcal{F})$ of $\mathcal{F}$ defines the corresponding lifting of frobenius to $\overline{N}_{D_y/\overline{Y},log}$. Note that

$$F_*({}_u\overline{c}_{\varphi(v)}) = {}_u\mathrm{par}_{P(\mathcal{F})(u)}\ \mathcal{F}_*({}_u\overline{c}_{\varphi(v)})\ {}_{\mathcal{F}(\varphi(v))}\mathrm{par}_{\varphi(v)},$$

where the *par* on the left and on the right denote the parallel transport along the connection on the spaces $\overline{N}_{D_y/\overline{Y},log}$ and on $\overline{Y}_{log}$ respectively. Note that since we are only interested in ${}_{\mathcal{F}(\varphi(v))}\mathrm{par}_{\varphi(v)}$ when v tends to zero we may assume that the parallel transport is in fact the parallel transport along the trivialized connection on $\varphi(\mathcal{U})_{log}$. In other words in the limit it can be replaced with $\prod_i exp(N_i \log \frac{\mathcal{F}(\varphi(v))_i}{\varphi(v)_i})$.

First note that since pulling back by $\mathcal{F}$ multiplies the residues of a connection with p, by the above limit computation for ${}_u\overline{c}_{\varphi(v)}$ we have

$$\lim_{v\to 0}\prod_i exp(-pN_i \log\frac{u_i}{v_i})\,\mathcal{F}_*({}_u\overline{c}_{\varphi(v)}) = 1.$$

Therefore

$$\lim_{v\to 0}\prod_i exp\big(-N_i \log\frac{u_i^p\, u_i\, \mathcal{F}(\varphi(v))_i}{v_i^p\, P(\mathcal{F})(u)_i\, \varphi(v)_i}\big)\, F_*({}_u\overline{c}_{\varphi(v)}) = 1.$$

Note that $P(\mathcal{F})$ is given by $P(\mathcal{F})(z_1,\cdots,z_r) = (a_1 z_1^p,\cdots,a_r z_r^p)$ for some $a_i \in 1+p\mathbb{Z}_p$. Since φ is identity on the tangent space $\lim_{v\to 0}\frac{\mathcal{F}(\varphi(v))_i}{v_i^p} = a_i = \lim_{v\to 0}\frac{P(\mathcal{F})(u)_i}{u_i^p}$, and $\lim_{v\to 0}\frac{v_i}{\varphi(v)_i} = 1$. And hence

$$\lim_{v\to 0}\prod_i exp\big(-N_i \log\frac{u_i}{v_i}\big)\, F_*({}_u\overline{c}_{\varphi(v)}) = 1,$$

and therefore $F_*({}_u\overline{c}_{\varphi(v)}) ={}_u \overline{c}_{\varphi(v)}$.

This gives the description of the canonical crystalline path between possibly tangential basepoints. If x and y have good reduction relative to a model then ${}_yc_x \in {}_y\mathcal{G}_{dR,x}(\mathbb{Q}_{p,st})$ is in fact defined over $\mathbb{Q}_p$. Again the argument cited above works in this case if we note that in the good reduction case in order to define the frobenius, by the above method of tangential basepoints, one does not have to tensor with $\mathbb{Q}_{p,st}$.

In order to see that changing the tangential basepoints by an $r-$tuple of roots of unity will have no effect in the crystalline de Rham theory we need the following lemma.

Lemma 10. *Let $\mathfrak{Y} := \mathbb{G}_m^r/\mathbb{Z}_p$, $\overline{\mathfrak{Y}}$ be the standard compactification and x and y be in $\mathfrak{Y}(\mathbb{Z}_p)$. Let*

$$\{e_i\}_{1\le i\le r} \in \pi_{1,dR}(Y,\omega(dR))$$

denote the residues at 0. Then ${}_{\omega(dR)}e(dR)_y F_*({}_ye(dR)_x)\,{}_xe(dR)_{\omega(dR)} = \prod_i exp(e_i \log\frac{y_i^{1-p}}{x_i^{1-p}})$.

Proof. We will assume without loss of generality that $x=1$ as the general statement above follows from this by concatenation of paths. Let $\mathcal{F}(z_1,\cdots,z_r) = (z_1^p,\cdots,z_r^p)$ this is a lifting to $\overline{Y}_{log}$ of the frobenius on the special fiber. Fix the beginning point as 1 and consider the $\pi_{1,dR}(Y,1)-$torsor of paths ${}_y\mathcal{G}_{1,dR}$ as y varies. Note that $\mathcal{F}_*$ defines a horizontal map $\mathcal{F}_* :{}_y \mathcal{G}_{1,dR} \to \mathcal{F}^*\,{}_y\mathcal{G}_{1,dR}$, where $\mathcal{G}_{dR}$ is endowed with its canonical connection. This gives a differential equation for $\mathcal{F}_*({}_ye(dR)_1)$ and solving this we find $\mathcal{F}_*({}_ye(dR)_1) = 1$, see [Ün] for a similar computation where more details are given. Since $F_*({}_ye(dR)_1) = {}_y\mathrm{par}_{\mathcal{F}(y)}\mathcal{F}_*({}_ye(dR)_1)$, as $\mathcal{F}(1)=1$ and ${}_y\mathrm{par}_{y^p} = \prod_i exp(e_i \log y^{1-p})$ the statement follows. Note that if y_i/x_i are roots of unity for all i then the above shows that $F_*({}_ye(dR)_x) = {}_ye(dR)_x$, as $\log(a)=0$ if a is a root of unity. $\square$

7.5. p-adic multi-zeta values.

Notation. For a smooth Y/K, and $y\in Y(K)$ let $\mathcal{U}_{dR}(Y,y)$ denote the universal enveloping algebra of $\mathrm{Lie}\pi_{1,dR}(Y,y)$ and $\hat{\mathcal{U}}_{dR}(Y,y)$ be its completion with respect to its augmentation ideal. It is a cocommmutative Hopf algebra and its topological dual is the Hopf algebra of functions on $\pi_{1,dR}(Y,y)$.

Letting e_0, e_1, and $e_\infty \in \mathrm{Lie}\pi_{1,dR}(X/\mathbb{Q}_p,t_{01})$ denote the residues corresponding to the points $0,1,\infty$ in $\overline{X}$ respectively, $\mathrm{Lie}\pi_{1,dR}(X,t_{01}) \simeq \mathrm{Lie} << e_0, e_1 >>$. $\hat{\mathcal{U}}_{dR}(X,t_{01})$ is isomorphic to the ring of associative formal power series on e_0 and e_1 with the coproduct Δ given by $\Delta(e_0) = 1\otimes e_0 + e_0\otimes 1$, and $\Delta(e_1) = 1\otimes e_1 + e_1\otimes 1$. By the duality above $\mathbb{Q}_p-$rational points of $\pi_{1,dR}(X,t_{01})$ correspond to associative formal power series a in e_0 and e_1 with coefficients in $\mathbb{Q}_p$, that start with 1 and satisfy $\Delta(a) = a\otimes a$.

We let

$$g :={}_{t_{01}}\, e(dR)_{t_{10}} F_*({}_{t_{10}}e(dR)_{t_{01}}) \in \pi_{1,dR}(X,t_{01}).$$

This is the series that defines the p-adic multi-zeta values. In particular the coefficient of the term $e_0^{s_k-1}e_1\cdots e_0^{s_1-1}e_1$ is by definition $p^{\sum s_i}\zeta_p(s_k,\cdots,s_1)$. These values also determine g.

2-cycle relation. Let $\gamma :={}_{t_{10}} e(dR)_{t_{01}}$, and we put $g := g(e_0, e_1) = \gamma^{-1} F_*(\gamma) \in \pi_{1,dR}(X_{\mathbb{Q}_p}, t_{01})$. We would like to see that

$$g(e_1, e_0)g(e_0, e_1) = 1.$$

Let τ be the automorphism of X that maps z to $1 - z$. Then

$$\tau(t_{01}) = t_{10},\ \tau_*(\gamma) = \gamma^{-1},\ \tau_*(e_0) = \gamma e_1 \gamma^{-1},\ \tau_*(e_1) = \gamma e_0 \gamma^{-1}.$$

We have

$$\gamma g(e_1, e_0)\gamma^{-1} = \tau_*(g(e_0, e_1)) = \tau_*(\gamma^{-1})\tau_*(F_*\gamma) = \tau_*(\gamma^{-1})F_*(\tau_*(\gamma)) = \gamma F_*(\gamma^{-1}).$$

Therefore

$$g(e_1, e_0) = F_*(\gamma^{-1})\gamma = g(e_0, e_1)^{-1}.$$

3-cycle relation. Let $\delta :={}_{t_{\infty 0}} e(dR)_{t_{01}}$, $r :={}_{t_{1\infty}} e(dR)_{t_{10}}$, and $q := r^{-1}\gamma ={}_{t_{1\infty}} e(dR)_{t_{01}}$. And let $e_\infty := \delta^{-1}\mathrm{res}_{t_{\infty 0}}\delta$ be the Lie element describing the residue at ∞ with basepoint t_{01}. Then we would like to see that

$$g(e_\infty, e_0)g(e_1, e_\infty)g(e_0, e_1) = 1.$$

Let ω be the automorphism of X that sends z to $\frac{1}{1-z}$. Then

$$\begin{aligned} \omega(t_{01}) &= t_{1\infty}, \text{ and } \delta = \omega_*(q)q,\ \omega_*(e_0) = qe_1q^{-1},\ \omega_*(e_1) = qe_\infty q^{-1}, \\ \omega_*^2(e_0) &= \omega_*^2(q)^{-1}e_\infty\omega_*^2(q),\ \omega_*^2(e_1) = \omega_*^2(q)^{-1}e_0\omega_*^2(q) \end{aligned}$$

Applying frobenius to

$$\omega_*^2(q)\omega_*(q)q = 1$$

we obtain

$$\begin{aligned} 1 &= \omega_*^2(F_*q)\omega_*(F_*q)F_*q = \omega_*^2(F_*q)\omega_*(F_*q)qq^{-1}F_*q \\ &= \omega_*^2(F_*q)\omega_*(q)q.\, q^{-1}\omega_*(q)^{-1}\omega_*(F_*q)q.\, g \\ &= \omega_*^2(F_*q)\omega_*(q)q.\, q^{-1}\omega_*(g(e_0, e_1))q.\, g = \omega_*^2(F_*q)\omega_*(q)q.\, g(e_1, e_\infty).\, g(e_0, e_1) \\ &= \omega_*^2(q)\omega_*^2(g(e_0, e_1))\omega_*^2(q)^{-1}.\, \omega_*^2(q)\omega_*(q)q.\, g(e_1, e_\infty).\, g(e_0, e_1) \\ &= g(e_\infty, e_0)g(e_1, e_\infty)g(e_0, e_1) \end{aligned}$$

since $r^{-1}F_*(r) = 1$, since $\log(-1)^{1-p} = 0$.

5-cycle relation. In this section we identify $M_{0,5}$ with $X^2 \setminus \{(z, w)|zw = 1\}$ by the map $(z, w) \to [0, z, 1, w^{-1}, \infty]$. By this identification let $t_{ij}(i, \epsilon)$ denote the tangent vector $(t_{ij}, 0)$ at the point (i, ϵ). And we choose, $t_{01,34}$ to be the tangent vector $(1, 1)$ at the point $(0, 0)$ and $t_{12,34}$ to be the vector $(-1, 1)$ at the point $(1, 0)$.

Lemma 11. *We have*

$$\lim_{y\to 0} exp(-e_{34}\log y)\ {}_{t_{01}(0,y)}c_{t_{01,34}} = 1,$$

where we use the canonical de Rham paths to identify the different basepoints.

Proof. By the description of the crystalline path with endpoints at a tangent vector we have, after always using the de Rham trivialization to identify the different basepoints,

$$\lim_{(x,y)\to(0,0)} exp(-e_{34}\log y)exp(-e_{01}\log x)\ {}_{(x,y)}c_{t_{01,34}} = 1$$

and

$$\lim_{x\to 0} exp(-e_{01}\log x)\ {}_{(x,y)}c_{t_{01}(0,y)} = 1,$$

which give the statement in the lemma. □

Similarly,

$$\lim_{y\to 0} exp(-e_{34}\log y)\ {}_{t_{10}(1,y)}c_{t_{12,34}} = 1.$$

Therefore we have

$${}_{t_{12,34}}c_{t_{01,34}} = \lim_{\epsilon\to 0} exp(-e_{34}\log\epsilon)\cdot\ {}_{t_{10}(1,\epsilon)}c_{t_{01}(0,\epsilon)}\cdot\ exp(e_{34}\log\epsilon).$$

Because of good reduction we know that the left hand side in fact is defined over $\mathbb{Q}_p$ therefore it is unchanged if we basechange by the map $\mathbb{Q}_{p,st} \to \mathbb{Q}_p$ that sends $l(p)$ to 0. We do this in the remaining part of the section. Therefore restricting ϵ to powers of p we obtain

$$_{t_{12,34}}c_{t_{01,34}} = \lim_{N\to\infty} {}_{t_{10}(1,p^N)}c_{t_{01}(0,p^N)}.$$

With the coordinates as above let X_ϵ be the subvariety of $M_{0,5}$ defined by $w - \epsilon = 0$, for $\epsilon \in \mathbb{Q}_p^*$. Therefore $X_\epsilon \simeq X \setminus \{1/\epsilon\}$. Note that $_{t_{01}(1,\epsilon)}c_{t_{01}(0,\epsilon)}$ is the image of the analogous element

$$c_\epsilon :=_{10} c_{t_{01}}(X_\epsilon)$$

under the inclusion $X_\epsilon \to M_{0,5}$.

We will need the following

Lemma 12. *Let a be a monomial in e_0, e_1 and $e_{p^{-N}}$ such that a contains an $e_{p^{-N}}$. If we denote the image of c_{p^N} under the canonical maps*

$$_{t_{10}}\mathcal{G}_{t_{01},dR}(X_{p^N}(\mathbb{Q}_{p,st}) \simeq \pi_{1,dR}(X_{p^N}, \omega(dR)) \subseteq \mathcal{U}_{dR}(e_0, e_1, e_{p^{-N}})(\mathbb{Q}_{p,st}) \simeq \mathbb{Q}_{p,st} << e_0, e_1, e_{p^{-N}} >>,$$

by the same symbol, then $\lim_{N\to\infty} c_{p^N}[a] = 0$.

Proof. First note that since c_{p^N} is a group-like element of $\hat{\mathcal{U}}_{dR}$ the coefficient $c_{p^N}[a]$ is a (finite) linear combination of terms $c_{p^N}[b]$ where b is a monomial that ends with $e_{p^{-N}}$. In order to see this first we write $a := a' \cdot a_0$, where a' ends with $e_{p^{-N}}$ and a_0 does not contain any $e_{p^{-N}}$. Then we compare the coefficients of the term $a' \otimes a_0$ on both sides of the equality $\Delta(c_{p^N}) = c_{p^N} \otimes c_{p^N}$, and use induction on the number of terms on the right of the last $e_{p^{-N}}$ in a.

Therefore without loss of generality we assume that a ends with $e_{p^{-N}}$. For $i \in \{0, 1, p^{-N}\}$, let $\omega_i := d\log(z - i)$. For a sequence of $\omega_i's : \omega_{i_r}, \ldots, \omega_{i_1}$ with $i_1 \neq 0$, we associate the iterated integral $I_{i_r,\cdots,i_1}(z_r) := \int_0^{z_r} \omega_{i_r} \circ \cdots \circ \omega_{i_1}$ defined successively by

$$I_{i_r,\cdots,i_1}(z_r) := \int_0^{z_r} \omega_{i_r}(z_{r-1}) I_{i_{r-1},\cdots,i_1}(z_{r-1}),$$

where by the misleading notation $\int_0^z$ we mean the anti-derivative of the integrand in the sense of the Coleman integral starting at the point 0. In the Coleman integration we let $\log p = l(p) = 0$, to be compatible with the above choice of a branch. This iterated integral is a locally analytic function on $X_{p^N,an}$. Note that since we assume that ω_{i_1} does not have a singularity at zero the iterated integral is in fact analytic around 0 with value 0 at 0. By definition of p-adic integration the iterated integral $I_{i_r,\cdots,i_1}(z)$ is the coefficient $_zc(X_{p^N})_{t_{01}}[e_{i_r} \cdots e_{i_1}]$. By the construction of Coleman integration

$$I_{i_r,\cdots,i_1}(z) = \sum_{0\le k\le n} f_k(z) \log^k(z-1)$$

on $U \setminus \{1\}$, where U is a neighborhood of 1 and f_k are analytic on U [Co].

By the definition of the tangential basepoints and the description above

$$_{t_{10}}c_{t_{01}}[a] = \lim_{z\to 1} \Big(exp(-e_1 \log(z-1)) \cdot {}_zc_{t_{01}} \Big)[a].$$

Therefore we obtain

$$_{t_{10}}c_{t_{01}}[a] = \lim_{M\to\infty} f_0(1 + p^M).$$

Therefore in order to prove the lemma it suffices to show that if $(\varphi_N(z))_N$ is a sequence of analytic functions on some $D(0, (1+\delta)^-)$ which converge uniformly to 0 with $\varphi_N(0) = 0$, then we have

$$\lim_{N\to\infty} \lim_{M\to\infty} \int_0^{1+p^M} \omega_{i_k} \circ \cdots \circ \omega_{i_1} \circ d\varphi_N(z) = 0.$$

We show this by induction on the weight, i.e. the number of terms in the iterated integral. The assertion is clear if the weight is one. Assume that the weight is greater than one. If $\omega_{i_1} = d\log(z - p^{-N})$ then by noting that

$$\int_0^z d\log(z - p^{-N}) \circ d\varphi_N(z)$$

satisfies the conditions for φ_N, we reduce to the case with one lower weight. The same is true if $\omega_{i_1} = d\log z$. So assume that $\omega_{i_1} = d\log(z-1)$. Note that

$$\int_0^z d\log(z-1)\circ d\varphi_N(z) = \int_0^z d\log(z-1)\cdot(\varphi_N(z)-\varphi_N(1)) + \varphi_N(1)\cdot\log(z-1)$$

The first term on the right satisfies the same conditions as φ_N therefore we only need to take care of the term $\varphi_N(1)\log(z-1)$. In other words need to show that

$$\lim_{N\to\infty}\lim_{M\to\infty}\varphi_N(1)\cdot\int_0^{1+p^M}\omega_{i_k}\circ\cdots\circ\omega_{i_1} = 0.$$

If there is a $d\log(z-p^{-N})$ among the ω_{i_j} this follows from the induction hypothesis, otherwise the iterated integral does not depend on N and $\lim_{N\to\infty}\varphi_N(1) = 0$ implies the assertion. $\square$

This lemma implies that ${}_{t_{12,34}}c_{t_{01,34}}$ only consists of the terms e_{01} and e_{12}. Let

$$\alpha := {}_{t_{01,34}}e(dR)_{t_{12,34}}\cdot{}_{t_{12,34}}c_{t_{01,34}}.$$

Then $F_*\alpha = ({}_{t_{12,34}}g_{t_{01,34}})^{-1}\cdot{}_{t_{12,34}}c_{t_{01,34}}$ and $F_*(\alpha)$ only consists of the terms e_{01} and

$$({}_{t_{12,34}}g_{t_{01,34}})^{-1}\cdot e_{12}\cdot{}_{t_{12,34}}g_{t_{01,34}}.$$

From this, by induction on the weight, we see that ${}_{t_{12,34}}g_{t_{01,34}}$ also consists only of e_{01} and e_{12}. By functoriality ${}_{t_{12,34}}g_{t_{01,34}}$ maps to ${}_{t_{10}}g_{t_{01}}$ under the map $M_{0,5}\to X$ and hence we have ${}_{t_{12,34}}g_{t_{01,34}} = g(e_{01},e_{12})$. Similarly we get

$$\begin{aligned}
{}_{t_{12,13}}g_{t_{12,34}} &= g(e_{34},e_{40})\\
{}_{t_{13,23}}g_{t_{12,13}} &= g(e_{12},e_{23})\\
{}_{t_{01,23}}g_{t_{13,23}} &= g(e_{40},e_{01})\\
{}_{t_{01,34}}g_{t_{01,23}} &= g(e_{23},e_{34})
\end{aligned}$$

Similar to the case of other relations we find that

$${}_{t_{01,34}}g_{t_{01,23}}\cdot{}_{t_{01,23}}g_{t_{13,23}}\cdot{}_{t_{13,23}}g_{t_{12,13}}\cdot{}_{t_{12,13}}g_{t_{12,34}}\cdot{}_{t_{12,34}}g_{t_{01,34}} = 1,$$

which gives the Drinfel'd-Ihara relation

$$g(e_{23},e_{34})\cdot g(e_{40},e_{01})\cdot g(e_{12},e_{23})\cdot g(e_{34},e_{40})\cdot g(e_{01},e_{12}) = 1.$$

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